Multiple Equilibria in Bidding Fee Auctions

by

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Abstract

This dissertation presents a game theoretic approach to bidding fee auctions with independent private values. I analyze these auctions under two bidding window rules. In a sequential bidding auction the round moves forward immediately after a bid was submitted. In a multiple round auction, the round moves forward only after all players have submitted their action. Under the assumption that the bidders may either have a low value or a high value for the object, I show that multiple equilibria, with relevantly different characteristics, may arise under either rule. Moreover, the rule that maximizes the seller's revenue depends on the the probability of a high value bidder.

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Whatever your work is, put your heart into it as done for the Lord and not for human beings. Col 3:23

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Contents

1. Introduction

A lot of controversy has arisen in the past few years due to the increasing popularity of bidding fee auctions, commonly known as penny auctions. These auctions are made online, with (potentially) many bidders struggling for a good (usually household objects, televisions, computers, gadgets, etc.). The first bidding fee auction site, Telebid (later changed its name to Swoopo) appeared in 2005, in Germany, and it was an immediate success: in the first four years of activity, its average revenues exceeded 150% of the goods retail price (Augenblick, 2011). However, such good results did not last long: Swoopo filed for bankruptcy in March 2011, due to financial difficulties. Even though, the number of penny auctions sites continued to grow and, currently, there are more than 500 allegedly penny auctions sites.

But how do these auctions work? A bidding fee auction usually has a timer between ten seconds and five minutes. When a bid is placed, the timer resets and the price goes up by a small amount (a *penny*). If no one bids before the end of the bidding window, the last bidder wins, paying the current high bid. In order to place a bid, bidders must pay a bidding fee.[1]

To illustrate how these auctions work, consider the auction of a television set. Suppose that, at a given moment, the current high bid is \$50.00. For someone to outbid this value, she must bid \$50.01, which has a fixed cost of \$1.00. People have 15 seconds to bid. If no-one bids until the timer reaches zero, the current high bidder will be the auction's winner, paying \$50.00 for the television set. This would give a \$5050 revenue[2] for the seller.

The fact that a bidder can spend a lot of money in the auction and still lose it and the extremely high average revenues for the auctioneer brought this new type of auctions to the attention of the media, comparing penny auctions to online gambling (McCarthy, 2011). The criticism increased when it was discovered that some sites were *shill bidding*, artificially increasing prices either manually or through bot bidders in the auction script. PennyBiddr, another penny auctions site, was shut down by the Washington State in 2010.[3] Besides

[1]The bidding credits are bought in bundles of bids. Usually, the unit cost of a bid is lower for larger bundles. In my model, I will ignore this, and assume the bidding marginal cost is constant.

[2]The final price, \$50.00, plus all the bidding revenues throughout the auction: $\frac{\$50.00}{\$0.01/bid} \times \$1/bid = \$5000$.

[3]See http://www.atg.wa.gov/pressrelease.aspx?id=26506.

shill bidding, PennyBiddr was also allowing its programme to actually win the auction, making it impossible for bidders to get the goods. The company was keeping both the money spent in bids by the real bidders and the object, which would be auctioned again.

Although there are no published papers on bidding fee auctions, several authors have already been working on the topic. Byers et al. (2010) provide the most complete analysis of penny auctions. They solve two general models: a model of ascending price and a model of fixed price[4] and extend their analysis to an auction environment in which the number of bidders is uncertain. They also model other types of asymmetries among the players: asymmetry in the bidding cost (some bidders may have bought a greater bundle of bidding credits, at a lower unit price), concluding that the existence of a group of bidders with lower marginal bidding cost increases the expected profits of the auctioneer; and asymmetry in the perceived common value of the item.[5] They further include shill bidding in the model, which unsurprisingly lowers the probability of entering the auction and, consequently, the expected seller's revenue; and they also let an aggressive bidding behaviour[6] by the bidders. Augenblick (2011) presents a theoretical model and finds its Markov perfect equilibrium, in terms of hazard rates, but focuses mainly on an empirical analysis, using data from Swoopo. On the demand side, he shows that bidders tend to bid more, the more financially committed they are to the auction (due to past bids), falling for the sunk-cost fallacy. On the supply side, his paper concludes that entrants tend to supply more auctions than it would be optimal in the short term, in order to create a large user-base. This creates barriers to entry and justifies why the profit margin is much higher for the market leader. Mittal (2010) provides a theoretical model, also with the assumption that a bidder's bidding decision depends on the amount invested in bid costs. She gives the conditions for the existence of symmetric equilibria and concludes that for sufficiently high bidding fees, a bidding fee auction can generate higher revenues, while for bidding fees approaching zero, the equilibrium would tend to the English auction's. She also states that the seller should expect higher revenues in excess of the object's

[4]In this model, the price is fixed (i.e., there is no price increment).

[5]The last one approaches what I do, independently, in this dissertation, with the major difference that they assume that a bidder perceives all rivals as having the same valuation as herself.

[6]A bidder using an aggressive bidding strategy bids immediately whenever she is outbid, to create a reputation that she will win the auction, no matter what.

retail price for objects the value of which is some orders of magnitude greater than the bidding fee. Platt et al. (2012) find that risk preferences play the most important role in bidding behaviour and stress the similarities between penny auctions and some forms of gambling. Because of that, this auction's format requires the most attention from the regulatory authorities. Hinnosaar (2010) underlines the unpredictability of revenues in symmetric stationary subgame perfect equilibria (players condition their actions on the current price and number of active bidders, instead of considering the whole history of bids), both for standard penny auctions (with a strictly positive price increment) and for auctions with zero price increment. The paper states that the variance of outcomes is a characteristic of the auction format.

The high variance of outcomes, that has also been pointed out in the media (McCarthy, 2011), motivates me to look for the existence of multiple equilibria in bidding fee auctions. My dissertation describes a bidding fee auction model of two players with independent private values. Byers et al. (2010), Hinnosaar (2010), Mittal (2010), Augenblick (2011), Platt et al. (2012), all present common value models, claiming that even though buyers may value the good differently, they all have the retail price of the good as reference. I consider that there can be bidders whose value is lower than the market price, and even though they would not buy the good elsewhere, they might be interested in participating in a penny auction to try to purchase it at a lower price. By using a private values' model, I allow the existence of two groups of bidders: bidders whose values are greater than the retail price of the good and bidders whose values are lower.

Another point where my work differs from the other authors' is that I assume leader proactivity, where they all assume leader passivity, meaning that the owner of the current high bid is unable to bid. The leader passivity assumption makes it simpler to look for symmetric equilibria. In my simple model, with only two bidders, I am able to loose that assumption without major complications. In an independent private values framework, it makes sense allowing the leader to bid, since she may learn about her rivals' values from observing their actions in the preceding round.

In reality the price should move up immediately after someone bidding. However if bidders bid almost simultaneously, they may not be able to instantaneously become aware of the price increase. Because of that, I provide two variations of the same model, that differ on the bidding window rule. The sequential bidding auction admits that as soon as a bid is placed, bidders acknowledge that they are in a different round. The multiple round bidding auction assumes that bidders are not able to instaneously see their rival's action. An auction with this rule is made of several rounds, in each of which all players can submit a bid. A round will only give place to the following by the end of a given time period.

In this dissertation, it is my aim to answer to the following questions: *Can bidding fee auctions have multiple perfect Bayesian equilibria?*, *Which bidding window rule maximizes the seller revenue?* and also *Is it possible that bidding fee auctions outperform standard auctions?*

The dissertation is organized as follows: In Section 2, I describe a two-player bidding fee auction model with independent private values. In Section 3, I solve the sequential bidding auction. In Section 4, I solve the multiple round bidding auction. In these two sections, besides deriving the equilibrium conditions, I construct a parametrized example. Section 5 compares both auctions' outcomes and shows what would be optimal, from the seller's point of view, if he could change the bidding window rule. I also compare the results of these models with the ones of an English auction benchmark, in terms of seller's profit. Finally, I use Section 6 to conclude and discuss further research in bidding fee auctions.

2. The Model

Consider a simple set-up with two potential buyers (1 and 2) and a passive auctioneer, who intends to sell an object.[7] The buyers value independently the item: each has a valuation $v^i \in \{v_L, v_H\}$, $i = 1, 2$, with $0 < v_L < v_H$, which can be seen as the monetary utility she[8] derives from consuming the object.[9] Initially, each v^i is private knowledge of bidder i, who has only an expectation θ_1 on the rival's type.

In this model, a round is a bidding decision moment for a given price. That is, at any round $t \in \{1, 2, ...\}$, each bidder must choose an action out of the set of possible actions $b_t^i = \{0, 1\}$, choosing $b_t^i = 1$ if she wants to bid or $b_t^i = 0$ if she would rather wait.

[7]For the sake of simplicity, I will assume the seller has zero reserve utility.

[8]Throughout the dissertation I will refer to each bidder as a "she" and to the seller as a "he".

[9]The existence of two different values can be interpreted as if there were two groups of bidders: some who value the good less than the retail value (v_L) and others that, even though they value more, are only willing to pay up to the market price (v_H). v_H would be the object's market price.

At any round $t \geq 2$, a player will either be or not be on the lead. I define the two possible states that bidders can hold throughout the auction:

Definition 1. The leader in a given round $t \geq 2$[10] is the owner of the current high bid.

Definition 2. A follower is any non-leader.

The state of a bidder in a given round depends on the actions taken by all bidders in the preceding round. If a single bidder decides to bid at t, she will become the leader at $t + 1$ for sure, while her rival will become the follower. However, if both decide to bid ($b_t^i = 1 \, \forall i \in \{1, 2\}$), the leader in the subsequent round will be randomly chosen in a move by Nature. After no-one bidding ($b_t^i = 0 \, \forall i \in \{1, 2\}$), the auction terminates and the current leader becomes the auction's winner. There is a non-refundable bidding fee $c > 0$ that must be paid by the bidder whenever she submits a bid. The reserve price is assumed to be zero and the price increment to be $\varepsilon > 0$.

In the following sections I present two variations of the model. The difference between the two is the rule that the auctioneer uses for the bidding window. The two rules are the following:

1. Sequential bidding auction: the bidding window has a maximum bidding period.
2. Multiple round bidding auction: the bidding window has a fixed bidding period.

In the first case, bidders have a time frame for submitting their bids. The first one submitting a bid pays the bidding fee and becomes on the lead in the following round. I call this rule "sequential bidding" because bidders have a maximum time to place a bid, but as soon as a bid is submitted, the round moves forward (the first one bidding becomes the following leader). In the multiple round bidding variation, there are multiple successive fixed time frames to bid. The round will only move forward by the end of the current bidding window (the last one bidding will be the following leader). In both cases, if no one bids before the countdown clock reaches zero, the auction closes and the current leader wins.

[10] In the first round there is no leader nor follower: both bidders have the same neutral state. In fact, at $t = 1$, all bidders could be seen as followers and the seller as the leader since if no one bids he will keep the object.

In the literature, these two rules have also been used. Hinnosaar (2010) and Mittal (2010) define round as a moment in time in which *all* non-leading bidders can either bid or wait. By their definition, the object's price can increase by $N\varepsilon$ units in a single round, if N bidders decide to bid in that round (multiple round bidding). Differently, Byers et al. (2010), Augenblick (2011) and Platt et al. (2012) use a bidding window definition that resembles mine of a sequential bidding auction.

These two hypothesis do not depart from reality and can illustrate different bidding environments. A framework in which bidders react rapidly, being able to instantaneously learn their rival's action, would be equivalent to the sequential bidding auction: immediately after the fastest player submitting a bid, her rival (who could also be willing to bid) perceives that the price has increased (*i.e.*: that they are in a new round) and makes a new bidding decision (for the new price). A sequential bidding auction could also be an approximation to a bidding fee auction with a long bidding window: since bidders have much time to bid, the probability of almost simultaneous bids is very close to zero, so bidders would have time to react to a rival's action. On the other hand, a framework in which bidders are slow-reacting, being unable to promptly see a rival's bid - either because the internet connection is slow or because bidders bid almost simultaneously (due to a short bidding window, for instance) - would be equivalent to the case of a multiple round bidding auction.

In both cases, I ignore the existence of a timer: in Appendix A.1 and Appendix B.1 I prove that introducing a countdown clock is unnecessary: it would only complicate the model without making it more realistic.

A game will be defined by 5-tuple $(c, \varepsilon, v_H, v_L, \theta_1)$, where c and ε are chosen by the auctioneer, v_H and v_L are bidders' characteristics and θ_1 is a probability distribution function. In the following two sections I will describe the equilibrium conditions for both the sequential bidding and the multiple round bidding auctions.

3. Sequential Bidding Auction

This framework is the one which is, theoretically, supposed to happen in reality. In a sequential bidding auction, the round moves forward immediately after a bid is submitted. Hence, there is certainty regarding the item's price in a given round: in round t, the high bid is $(t-1)\varepsilon$. I assume that if more than one bidder chooses to bid, Nature will randomly select the following leader, and only her will have to pay the bidding fee.[11] The price increases by ε in each round.

In this section I present an equilibrium analysis of this model and then I solve a parametrized example.

3.1. Equilibrium analysis

In any round, a bidder will choose the action that maximizes her expected pay-off.

Let:

$s_t^i \in \{f, l\}$ be the state of bidder i in round $t \geq 2$.
$s_t^i = l$ if bidder i is the leader at t.
$s_t^i = f$ if bidder i is a follower at t.

Let also $u_t^i(v^i, s_t^i)$ be the expected utility for bidder i in round t, if her state is s_t^i .

The game matrix in round t is summarized in Table 1:

Bidder i's action	Rival bids ($b_t^j = 1$)	Rival waits ($b_t^j = 0$)
bid ($b_t^i = 1$)	$0.5\Big(u_{t+1}(v^i, l) - c\Big) + 0.5\Big(u_{t+1}(v^i, f)\Big)$	$u_{t+1}(v^i, l) - c$
wait ($b_t^i = 0$)	$u_{t+1}(v^i, f)$	ue_t^i

Table 1: Game matrix in round t

where ue_t is the pay-off bidder i will get if the game closes by the end of round t. Its value depends on the state of the bidder:

$$ue_t^i(s_t^i) = \begin{cases} v^i - t\varepsilon & if\ s_t^i = l \\ 0 & if\ s_t^i = f \vee t = 1 \end{cases}$$

The fact that, when a buyer successfully bids, she pays a non-refundable fee c plays an important role in the equilibrium. Since c is sunk in each round, the amount that was spent in previous bids is not taken into consideration in current and further bidding decisions.[12] Nevertheless, the current price and the

[11] This may seem unrealistic but it ends up being a good approximation with reality. The two bidders may have decided to bid, but one was faster than the other placing the bid, overcoming the rival. Nature plays the role of choosing the fastest bidder.

[12] Other authors, as Augenblick (2011) and Mittal (2010), include the amount invested in bid costs in the bidders' utility function.

cost of placing an additional bid do matter. We identify two exit rounds, which will be useful to find the equilibrium strategies of the game.

Proposition 1. *In equilibrium, a bidder of type x does not consider bidding in any round $t > \frac{v_x - c}{\varepsilon}$.*

Proof. Let $b^j_{t+1} = 0$. At $t+1$, a leader i of type x would win the auction and get a gross utility of $v_x - t\varepsilon$. In the previous period, t, a buyer bids if and only if she finds profitable to attempt to be on the lead in the following round, incurring in an additional bidding fee. Then, $v_x - t\varepsilon - c < 0$ is a sufficient condition for a bidder of type v_x not to bid. □

Definition 3. $\underline{T} \equiv \lfloor \frac{v_L - c}{\varepsilon} \rfloor + 1$ and $\overline{T} \equiv \lfloor \frac{v_H - c}{\varepsilon} \rfloor + 1$ are the exit rounds for low valuation and high valuation bidders, respectively.[13]

It is straightforward noticing that after $t = \underline{T}$ all the observed bids must come from high value bidders. Moreover, if $t = \overline{T}$ is ever reached, we are sure that the auction will end by the end of that period.[14] Therefore, I can start the backwards analysis at $\overline{T}$, in order to find the equilibrium strategies of the game.

Since there is uncertainty regarding the valuation of the other bidder, the extensive form of this game is one of incomplete information.[15] I will use the concept of perfect Bayesian equilibrium, specifying a probability distribution (belief) for each player over the nodes in all information sets: in a given information set, each player must have a belief about the rival's type. For a perfect Bayesian equilibrium to exist, the players' actions must be optimal given their beliefs and the rival's strategy (Gibbons, 1992).

In my model, each player's initial belief (i.e.: the ex-ante probability of a high value bidder) is a probability mass function, where $\Pr(v_i = v_H | t = 1) \equiv \theta_1$ and $\Pr(v_i = v_L | t = 1) \equiv 1 - \theta_1$, for $i = 1, 2$. [16] I consider the case of a symmetric auction: both player's types are drawn from the same distribution function.

[13] $\lfloor \cdot \rfloor$ is the floor function. $\lfloor \alpha \rfloor$ gives the greatest integer lower or equal than α, for any $\alpha \in \mathbb{R}$.

[14] At $t = \overline{T}$ it will not be profitable for anyone to bid. This means that the leader at $\overline{T}$ will be the auction's winner, getting a pay-off of $v_H - (\overline{T} - 1)\varepsilon$, net of all bidding costs incurred in the previous rounds.

[15] This dynamic game of incomplete information can be converted to a dynamic game of complete but imperfect information (Nature chooses the *types* of the bidders in the beginning of the game, according to the probability distribution θ_1). Then we use the subgame perfect equilibria of this game as the solution concept.

[16] θ_1 can also be interpreted as the share of bidders, in the universe of potential buyers, for whom the object has a valuation v_H.

Given the current state of bidder j,[17] bidder i updates her belief about bidder j's type through Bayes' rule.[18]

Let σ_k^i be the vector of the $k-1$ states of bidder i from $t=2$ to $t=k$, i.e. $\sigma_k^i = (s_2^i, .., s_{k-1}^i, s_k^i)$.

$$\theta_t^j(\sigma_t^j) = \Pr(v^j = v_H|\sigma_t^j) =$$
$$= \frac{\Pr(s_t^j|v^j = v_H) \cdot \theta_{t-1}^j}{\Pr(s_t^j|v^j = v_H) \cdot \theta_{t-1}^j + \Pr(s_t^j|v^j = v_L) \cdot (1-\theta_{t-1}^j)} \; for\, t \geq 2 \qquad (1)$$

Definition 4. A belief system θ of this model assigns beliefs to all possible nodes in all information sets.

In spite of choosing $b_t^i \in \{0,1\}$, bidders may bid with a given probability. We can redefine the decision of bidder i to a probability of bidding.

Let $\rho_t^i(\sigma_t^i) = \rho_t^i(s_t^i, \sigma_{t-1}^i) \equiv \Pr(b_t^i = 1|v^i \wedge \sigma_t^i)$, with $v^i \in \{v_H, v_L\}$ for $t \geq 2$.[19]

Let $\rho_1^i \equiv \Pr(b_1^i = 1|v^i)$.

Let $\rho_t^x(\sigma_t^i)$ be the mixed strategy equilibrium probability of a bidder of type v_x bidding in round t.

Proposition 2. *The Bayesian updating equation depends on the current state of bidder j:*

$$\theta_t^j(s_t^j = f, \sigma_{t-1}^j) = \frac{(2-\rho_{t-1}^H) \cdot \theta_{t-1}^j}{(2-\rho_{t-1}^H) \cdot \theta_{t-1}^j + (2-\rho_{t-1}^L) \cdot (1-\theta_{t-1}^j)} \qquad (2)$$

$$\theta_t^j(s_t^j = l, \sigma_{t-1}^j) = \frac{\rho_{t-1}^H \cdot \theta_{t-1}^j}{\rho_{t-1}^H \cdot \theta_{t-1}^j + \rho_{t-1}^L \cdot (1-\theta_{t-1}^j)} \qquad (3)$$

Proof. At any round $t \geq 2$, bidder j can be the follower either because she waited or because she bid but Nature chose bidder i to be the leader instead. However, if bidder j is the follower and the game is still active, bidder i must

[17]Since there are only two bidders and two possible states, conditioning the beliefs on the state of bidder i or bidder j is the same.

[18]Beliefs both on and off the equilibrium path must be updated through Bayes' rule whenever possible. $\theta_t^j(s_t^j, ...s_1^j)$ is the belief, for player i in round t, that player j is of the high type, given all the past states of bidder j. For nodes that are reached with zero probability Bayes' rule cannot be used. In those cases any belief is admissible.

[19]To make notation simpler, throughout the paper I will sometimes denote $\rho_t^i(\sigma_t^i)$ as ρ_t^i.

have bid. Therefore, the probability of bidder j being the follower given she is of type $v_x \in \{v_H, v_L\}$ must be given by:

$$\begin{aligned}\Pr(s_t^j = f | v^j = v_x \wedge b_t^i = 1) =& \Pr(b_{t-1}^j = 0 | v^j = v_x) + 0.5 \Pr(b_{t-1}^j = 1 | v^j = v_x) \\ =& 1 - 0.5\rho_{t-1}^x(\sigma_{t-1}^j)\end{aligned}$$

Replacing this expression in Equation 1 gives us Equation 2.

Conversely, bidder j can only be the leader if she bid at $t-1$. The probability of bidder j being the leader given she is of type v_x depends on the action of bidder i:

$$\begin{aligned}\Pr(s_t^j = l | v^j = v_x \wedge b_t^i = 0) =& \Pr(b_{t-1}^j = 1 | v^j = v_x) \\ =& \rho_{t-1}^x(\sigma_{t-1}^j)\end{aligned}$$

$$\begin{aligned}\Pr(s_t^j = l | v^j = v_x \wedge b_t^i = 1) =& 0.5 \Pr(b_{t-1}^j = 1 | v^j = v_x) \\ =& 0.5\rho_{t-1}^x(\sigma_{t-1}^j)\end{aligned}$$

Replacing either of these two expressions in Equation 1 gives us Equation 3. □

Let ρ be any strategy profile of the game, specifying an action for each player's type and round. A perfect Bayesian equilibrium of this game will be given by a strategy profile ρ^*, which is optimal given a system of beliefs θ^* and the rival's strategies. Let also $V^i(\rho|\theta)$ be the expected pay-off of bidder i for strategy profile of the game is ρ and belief system θ.

Each player's strategy (ρ^i) must include an action for all possible nodes in the game. A v_L player's strategy must have an action for each possible situation in rounds $t \in \{1, \underline{T}\}$, which means that her strategy will be made of $\sum_{t=1}^{\underline{T}} 2^{t-1}$ actions[20]. Similarly a v_H player's strategy must include $\sum_{t=1}^{\overline{T}} 2^{t-1}$ actions.

Definition 5. There is a perfect Bayesian equilibrium of the game if:

[20] At $t = \underline{T}$ a v_L does not bid ($b_t^i = 0$), regardless of the information set. The same applies for v_H bidders at $t = \overline{T}$.

$$V^i(\rho^*|\theta^*) \geq V^i(\rho_i, \rho^*_{-i}|\theta^*)\, \forall i \in \{1,2\}$$

and θ determined by Bayes' rule when possible.[21]

I look at two types of perfect Bayesian equilibria: symmetric and asymmetric. Symmetric equilibria are equilibria in which bidders with the same valuation and with the same information set bid similarly. The first round actions must be $\rho_1^i = \rho_1^H\, \forall v^i = v_H$ and $\rho_1^i = \rho_1^L\, \forall v^i = v_L$, with $\rho_1^H \in [0,1]$, $\rho_1^L \in [0,1]$. No-one is willing to deviate as long as they believe a rival of the same type is bidding similarly and a rival of the other type (say type $-x$) is playing ρ_1^{-x}.

On the contrary, in asymmetric equilibria, bidders with the same information set bid differently. In this set-up bidders can only have the same information set in the beginning of the first round. From round two onwards, their states differ, meaning that they have different information sets. The first round equilibrium actions are $\rho_1^i = 0$ for $\rho_1^j = 1$ and $\rho_1^i = 1$ for $\rho_1^j = 0$, independently of the bidders type. That is saying that a bidder is only willing to bid if she believes her rival will not, and vice-versa. Note that there may be other types of asymmetric equilibria, but I only focus on this one.

Next, I illustrate this model with a simple parametrized example. This example does not strive to mimic an actual auction of a real object, but to show that different equilibria may arise.

3.2. An example

Consider a parametrized family of examples of the previous model, with $0 \leq \theta_1 \leq 1$, where bidders may have a valuation of either 2.5 or 4.5.[22] Each bid makes the price increase by a unit and the bidding fee is also unitary[23] (Table 2).

For these specific values of the parameters, equilibria with the strategy profiles described in Table 3 may arise. It is not worth specifying v_L's actions for $t = \underline{T}$ nor v_H's actions for $t = \overline{T}$: we are sure they *wait* in those rounds (Proposition 1).

Table 3 includes all symmetric equilibria for any initial belief. Asymmetric equilibria cannot be reached for the parameters values in this example.[24] Figure

[21]See Appendix A.2.
[22]$\underline{T} = 2$ and $\overline{T} = 4$.
[23]It is very uncommon to have a price increment equal to the bidding fee. I set $c = \varepsilon = 1$ for the sake of simplicity.
[24]See Proposition 5, in Appendix A.2.

Parameter		Value
bidding fee	c	1
price increment	ε	1
low value	v_L	2.5
high value	v_H	4.5

Table 2: Parametrization

		$t=1$		$t=2$		$t=3$			
	θ_1	ρ_1^L	ρ_1^H	$\rho_2^H(f)$	$\rho_2^H(l)$	$\rho_3^H(f,f)$	$\rho_3^H(f,l)$	$\rho_3^H(l,f)$	$\rho_3^H(l,l)$
$a)$	$[0,\frac{1}{3})$	1	1	1	0	1	1	0	0
$b)$	$[\frac{1}{3},\frac{3}{7})$	$\frac{2-4\theta_1}{1-\theta_1}$	1	1	0	1	1	0	0
$c)$	$(\frac{3}{5},\frac{8}{9})$	1	1	0	0	1	1	0	0
$d)$	$(\frac{5}{6},\frac{16}{17})$	0	0	1	0	1	1	0	0
$e)$	$(\frac{8}{9},1]$	1	1	0	0	1	1	1	1
$f)$	$(\frac{16}{17},1]^*$	0	0	1	0	1	1	0	1

(*) Equilibrium f) needs the out of equilibrium belief $0 \leq \theta_2^j(l) < \frac{3}{5}$.
Note: At $t = \underline{T} = 2$, $\rho_2^L(s_2) = 0\,\forall s_2 \in \{l, f\}$ and at $t = \overline{T} = 4$, $\rho_4^H(\sigma_4) = 0\,\forall \sigma_4$.

Table 3: Equilibrium strategy profiles for different initial beliefs in a sequential bidding auction.

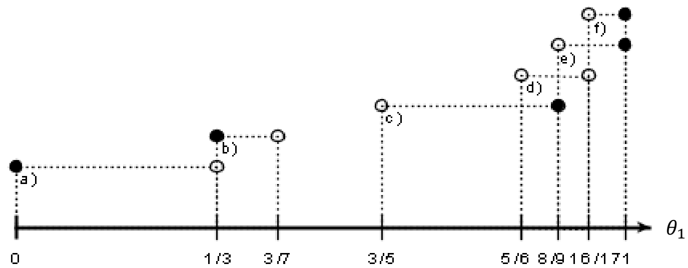

Figure 1: Perfect Bayesian equilibria for each initial belief in a sequential bidding model.

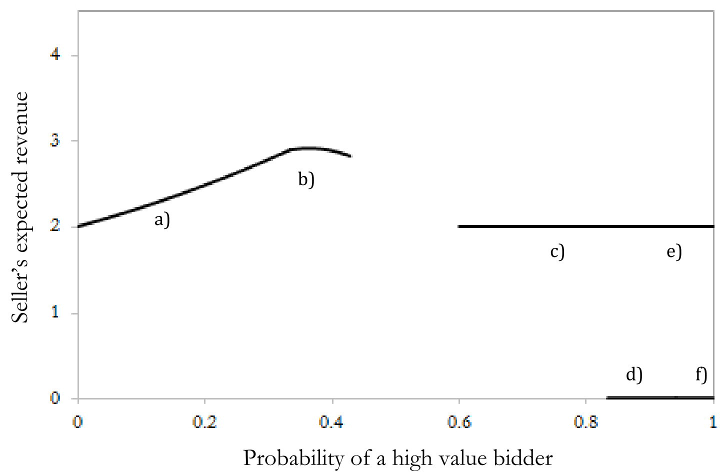

Figure 2: Seller's expected revenue for all possible symmetric equilibria in a sequential bidding model.

1 includes the intervals of θ_1 for which equilibria $a)$ to $f)$ exist.

It is readily verifiable that there is at least a perfect Bayesian equilibrium for each initial belief, except for beliefs in the interval $[\frac{3}{7}, \frac{3}{5}]$,[25] where there are none. For $\theta_1 \in (\frac{5}{6}, 1]$ *two* equilibria co-exist, one in which everyone bids, independently of the type, and another where bidders prefer not to start the auction. Different equilibria lead to different revenues: the expected revenue for the seller is closely dependent on the initial expectation, as it is shown in Fig. 2.

Let us consider a fully parametrized example, with $\theta_1 = 0.85$. Under this initial belief we can have equilibria fitting in $c)$ and $d)$. I will characterize the

[25] By Nash's theorem one can be sure that at least a Nash equilibrium exists.

	Pr. of win.		Expected return			
Eq.	v_L	v_H	v_L	v_H	seller	SW
$c)$	0.5	0.5	0.25	1.25	2	4.2
$d)$	0	0	0	0	0	0

Table 4: Player's probability of winning and expected pay-offs in the two equilibria under $\theta_1 = 0.85$.

equilibria outcomes in terms of perceived (ex-ante) probability of winning and expected utility for bidders of both types, seller's revenue and social welfare.[26] These results are in Table 4.

In equilibrium $d)$ the object is not sold, leading to zero pay-offs for all interested parties. In equilibrium $c)$, however, the object is sold for sure and the seller (with zero reserve utility) has a certain pay-off. The bidders' pay-off is uncertain, since it depends on the unpredictable action of Nature.

Even though equilibrium $d)$ is Pareto superior to equilibrium $d)$ (all risk neutral agents are better off), anonymous bidders may not be able to cooperate to choose one equilibrium over the other. If they were, the seller would encourage that behaviour.

4. Multiple Round Bidding Auction

Section 3 described a model where the successful bidder of a given round is the first to bid. However, in reality, it can happen that, for some reason (short bidding window, almost simultaneous bids, low internet connection, etc.), bidders are unable to promptly realize that a rival has placed a bid. Therefore, several bids can be placed in the same round and uncertainty is brought in what concerns the price of the object in the following round.[27] In such a framework, the successful bidder is the last one bidding in a round.

In this section I provide a model that tries to mimic this effect. The auction is made of consecutive fixed time rounds, where any bidder can submit a bid.

[26] The social welfare indicator I use is the following: $E[SW] = E[SR] + \sum_y \sum_z \prod_x \Pr[v^i = v_x]\Big(E[V^y] + E[V^z]\Big)$, for $y, z = \{L, H\}$ and $x = \{y, z\}$

[27] This effect is observed in penny auctions houses, when bidders concentrate their bids in the ending seconds of a round. When several bidders bid almost simultaneously, rounds move very quickly (the timer is restarted several times in a row), but because the change is so fast, bidders perceive a unique (yet bigger) increase in the price.

Thus, whenever a player decides to bid, she will have to pay the bidding cost even if she does not become on the lead in the following round: the leader in round t is randomly selected from the set of bidders who did bid at $t-1$.[28] In the previous framework, the high bid in round t was $t\varepsilon$. Now, *a priori*, we only know that the price in the beginning of round t, p_t, will be between $(t-1)\varepsilon$ and $2(t-1)\varepsilon$, depending on the actions taken by the bidders in all previous rounds.[29]

One could argue that the existence of a timer does matter in this framework, stating that a bidder (who wants to submit a bid) prefers to wait until the last tenth of a second of the round, in order try to be the last one bidding in that round. However, if all bidders behave like this, the way I describe the model, ignoring the timer, has a good fit with reality. In Appendix B.1 I prove that using a timer in the model is indeed irrelevant.

4.1. Equilibrium analysis

The exit rounds for v_H and v_L bidders are as in Definition 1.

Let β_t^i be the vector of actions of bidder i, from round 1 to t; $\rho_t^i(s_t^i, p_t)$ be the mixed strategy probability of bidder i bidding in round t, given her current state and the current price; and μ_1 be the initial probability of a high value bidder.[30]

In the subsequent rounds ($t \geq 2$), bidders update their beliefs through Bayesian updating:

$$\mu_t^j(\beta_{t-1}^j) = \frac{\Pr[b_{t-1}^j | v^j = v_H] \cdot \mu_{t-1}^j}{\Pr[b_{t-1}^j | v^j = v_H] \cdot \mu_{t-1}^j + \Pr[b_{t-1}^j | v^j = v_j] \cdot (1 - \mu_{t-1}^j)} \quad (4)$$

Since actions were partially unobservable[31], in the previous section, the updating rule depended on the state of the follower. In a multiple round bidding auction model, actions are observable by the end of the round, so the updating rule depends on the rival's previous action.

[28] Now Nature plays the role of choosing the last one bidding in a round.

[29] If we are in round $t-1$, we know that p_t will be between p_{t-1} (if no-one bids at $t-1$) and $p_{t-1} + 2\varepsilon$ (if both bid).

[30] I use μ_1 rather than θ_1, as in the previous framework, to point out that the belief system of this model is different (it depends on a vector of actions instead of depending on a vector of states), despite the meaning of the initial beliefs μ_1 and θ_1 being exactly the same.

[31] If two bidders decided to bid, the one who ended up being the leader in the following round was not able to know if her rival had or had not also decided to bid.

Proposition 3. *The Bayesian updating equation depends on the (visible) action of bidder j in round $t-1$:*

$$\mu_t^j(b_{t-1}^j = 0, \beta_{t-2}^j) = \frac{(1-\rho_{t-1}^H)\cdot\mu_{t-1}^j}{(1-\rho_{t-1}^H)\cdot\mu_{t-1}^j + (1-\rho_{t-1}^L)\cdot(1-\mu_{t-1}^j)} \tag{5}$$

$$\mu_t^j(b_{t-1}^j = 1, \beta_{t-2}^j) = \frac{\rho_{t-1}^H\cdot\mu_{t-1}^j}{\rho_{t-1}^H\cdot\mu_{t-1}^j + \rho_{t-1}^L\cdot(1-\mu_{t-1}^j)} \tag{6}$$

Proof. The equilibrium probability of a bidder of type v_x bidding in round $t-1$ is $\Pr[b_{t-1}^j = 1 | v^j = v_x \wedge s_{t-1}^j \wedge p_{t-1}] = \rho_{t-1}^x(s_{t-1}^j, p_{t-1})$. Conversely, the probability of waiting is $\Pr[b_{t-1}^j = 0 | v^j = v_H \wedge s_{t-1}^j \wedge p_{t-1}] = 1 - \rho_{t-1}^x(s_{t-1}^j, p_{t-1})$. Replacing these in Equation 4, gives us Equation 5 and Equation 6, respectively. □

The condition for the existence of a perfect Bayesian equilibrium is as follows:

Definition 6. There is a perfect Bayesian equilibrium of the game if:

$$V^i(\rho^*|\mu^*) \geq V^i(\rho_i, \rho_{-i}^*|\mu^*)\,\forall i \in \{1,2\}$$

and μ determined by Bayes' rule when possible.[32]

Once more, both symmetric and asymmetric equilibria can occur.

4.2. An example

In order to later compare this model to the sequential bidding one, I use the same parametrized example (Table 2).

In the multiple round auction, these parameter values allow for the existence of both symmetric and asymmetric equilibria. Table 5 includes all possible *symmetric* equilibria of the game: equilibria in which bidders with the same valuation and with the same information set bid similarly.

In the asymmetric equilibria of the game, a single potential buyer bids in the first period, independently of the valuations of both. In this example, asymmetric equilibria exist for $\mu_1 \in [0, \frac{1}{3})$ and $\mu_1 \in (\frac{3}{5}, 1]$ (Table 6).

Figure 3 shows that for more than sixty percent of the space of the initial belief $\mu_1 \in [0,1]$, multiple equilibria may occur. For $[0, \frac{3}{14}]$ and $(\frac{3}{5}, \frac{5}{6}]$ there are

[32]See Appendix B.2.

	μ_1	$t=1$		$t=2$		$t=3$	
		ρ_1^L	ρ_1^H	$\rho_2^H(f,p_2)$	$\rho_2^H(l,p_2)$	$\rho_3^H(f,p_3)$	$\rho_3^H(l,p_3)$
$a)$	$[0, \frac{3}{14}]$	$\frac{2-6\mu_1}{5-5\mu_1}$	1	1	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0
$b)$	$[\frac{2}{5}, \frac{2}{3}]$	$\frac{2-3\mu_1}{3-3\mu_1}$	1	$0\ if\ p_2 = 1$ $1\ if\ p_2 = 2$	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0
$c)$	$(\frac{2}{3}, \frac{10}{13})$	0	1	$0\ if\ p_2 = 1$ $1\ if\ p_2 = 2$	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0
$d)$	$[\frac{10}{13}, 1]$	0	$\frac{10}{13\mu_1}$	$0\ if\ p_2 = 1$ $1\ if\ p_2 = 2$	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0
$e)$	$(\frac{5}{6}, 1]$	0	0	1	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0

Note: At $t = \underline{T} = 2$, $\rho_2^L(s_2, p_2) = 0\, \forall s_2 \in \{l, f\}, \forall p_2$ and at $t = \overline{T} = 4$, $\rho_4^H(s_4, p_4) = 0 \forall s_4 \in \{l, f\}, \forall p_4$.

Table 5: Symmetric equilibrium strategy profiles for different initial beliefs, in a multiple round bidding auction.

	μ_1	$t=1$	$t=2$		$t=3$	
		ρ_1^i	$\rho_2^H(f,p_2)$	$\rho_2^H(l,p_2)$	$\rho_3^H(f,p_3)$	$\rho_3^H(l,p_3)$
$f)$	$[0, \frac{1}{3})^*$	$0\ if\ \rho_1^j = 1$ $1\ if\ \rho_1^j = 0$	1	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0
$g)$	$(\frac{3}{5}, 1]$	$0\ if\ \rho_1^j = 1$ $1\ if\ \rho_1^j = 0$	$0\ if\ p_2 = 1$ $1\ if\ p_2 = 2$	0	$0\, if\, p_3 = \{3,4\}$ $1\, if\, p_3 = 2$	0

$(^*)$ Equilibrium $f)$ also needs the out of equilibrium belief $0 < \mu_2^l(b_1^j = 1) \leq \frac{3}{5}$.
Note: At $t = \underline{T} = 2$, $\rho_2^L(s_2, p_2) = 0\, \forall s_2 \in \{l, f\}, \forall p_2$ and at $t = \overline{T} = 4$, $\rho_4^H(s_4, p_4) = 0 \forall s_4 \in \{l, f\}, \forall p_4$.

Table 6: Asymmetric equilibrium strategy profiles for different initial beliefs, in a multiple round bidding auction.

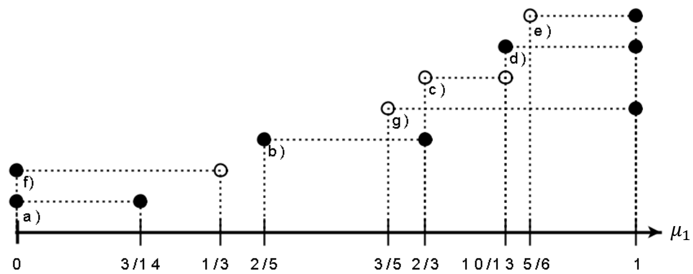

Figure 3: Perfect Bayesian equilibria for each initial belief.

two possible equilibria; and for $(\frac{5}{6}, 1]$ three equilibria co-exist. As in Section 3.2, there is also a region of μ_1 where there are no symmetric nor asymmetric equilibria: $[\frac{1}{3}, \frac{3}{5})$.

In the symmetric equilibria of this example, the object is always efficiently allocated to the bidder who most values it. The same is not true in asymmetric equilibria of type g): a v_H follower at $t = 2$ waits, anticipating that if she did bid and it was the case that her rival was also v_H (which is true with a very high probability), the latest would outbid her in the subsequent round, winning the auction. The auction's winner is the leader in the second round, independently of the type.

Figure 4 shows, for all possible equilibria, how does the seller's revenue vary with the initial belief. The continuous line represents the seller's revenue under symmetric equilibria, while the dashed line shows the asymmetric equilibria's revenues.

To show, in greater detail, that different (yet simultaneous) equilibria can have relevantly different results, not only in terms of revenue for the seller, but also in terms of bidders' pay-offs and social welfare, I use two fully parametrized examples.

Example 1. $\mu_1 = 0.2$.

The two equilibria that exist under $\mu_1 = 0.2$ have different outcomes (Table 7). Equilibrium f) provides superior expected pay-offs to the seller and to v_L bidders. On the contrary, a v_H bidder would rather be in an equilibrium of type a). The fact that a v_L is indifferent between bidding or waiting (mixing) in a), gives her a zero expected utility, even if she ends up winning the auction (which can only happen if her rival is also v_L). Although both equilibria ensure an

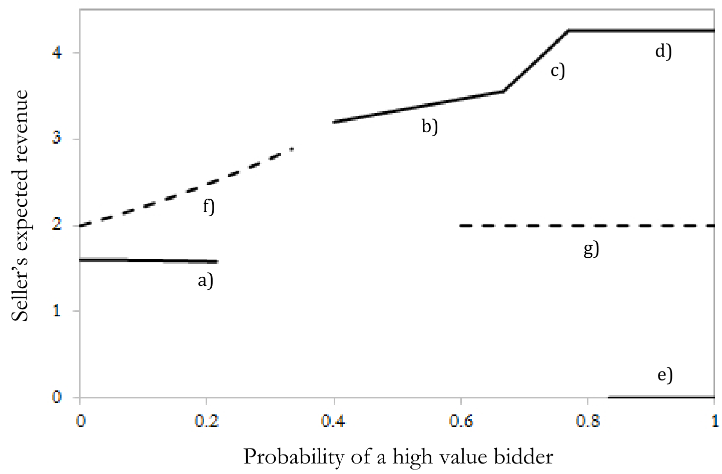

Figure 4: Seller's expected revenue for all possible symmetric (continuous line) and asymmetric (dashed line) equilibria in a multiple round bidding model.

	Pr. of win.		Expected return			
Eq.	v_L	v_H	v_L	v_H	seller	SW
$a)$	0.14	0.9	0	1.53	1.58	2.2
$f)$	0.4	0.9	0.1	1.45	2.48	3.22

Table 7: Probability of winning and expected pay-offs for the two equilibria that exist under $\mu_1 = 0.2$. The asymmetric equilibrium results are assuming that both possible equilibria ($\rho_1^1 = 1$, $\rho_1^2 = 0$) and ($\rho_1^1 = 0$, $\rho_1^2 = 1$) have equal probability of occurrence. This is a simplifying assumption to facilitate the comparison with the symmetric equilibrium.

	Pr. of win.		Expected return			
Eq.	v_L	v_H	v_L	v_H	seller	SW
$d)$	0	0.56	0	0	4.26	4.26
$e)$	0	0	0	0	0	0
$g)$	0.5	0.5	0.25	1.25	2	4.2

Table 8: Probability of winning and expected pay-offs for the three equilibria that exist under $\mu_1 = 0.85$. The asymmetric equilibrium results are assuming that both possible equilibria ($\rho_1^1 = 1$, $\rho_1^2 = 0$) and ($\rho_1^1 = 0$, $\rho_1^2 = 1$) have equal probability of occurrence. This is a simplifying assumption to facilitate the comparison with the symmetric equilibrium.

efficient allocation of the item, it is also $f)$ that leads to the greatest expected social welfare.

Example 2. $\mu_1 = 0.85$

With this initial expectation, three perfect Bayesian equilibria are possible (Table 8), including two symmetric: one where, at $t = 1$, only players with high valuation may bid, using a mixed strategy action $\rho_1^H = \frac{200}{221}$ (equilibrium of type $d)$) and another in which no-one bids (equilibrium of type $e)$); and an asymmetric equilibrium (type $g)$). The first one (type $d)$) is clearly the most desired equilibrium for the seller, giving him an expected return close to the high value. It is also the one that provides the greatest expected social welfare. The second (type $e)$) has the most inefficient allocation possible: the agent that least values the object (the seller) is the one who keeps it. The allocation of the item in the equilibrium of type $g)$ is random: the object will be sold, but the winner can either be a high value or a low value bidder. Although, efficiency in terms of allocation is not guaranteed, bidders of both high and low types are better-off in this equilibrium than in any of the remaining two. Even though the bidders' favourite equilibrium is equilibrium $d)$, society as a whole would be worse-off if this equilibrium was the one being played: not only it can lead to inefficient allocations, but also it is not the one that provides the greatest expected overall welfare.

Coordination between anonymous bidders who do not know each other is hardly plausible. However, in both examples, the bidders could gain if they were able to implicitly collude. Even if a bidder intentionally loses an auction by refraining from bidding, she could be compensated by her rival in further auctions. In Example 1, only bidders of equal types would agree to collude, making collusion even less probable of occurring. The seller would be interested in encouraging coordination by low value bidders, since they would choose the seller's favourite equilibrium. In Example 2, if bidders were able to implicitly collude, they would harm not only the seller, but society as a whole. Nevertheless, implicit collusion is very unlikely to happen as we introduce more bidders to the game.

5. Comparing the two rules

The two equilibria I derived in the previous sections try to describe the behaviour of bidders under different bidding window rules. Even though it can be argued that, in reality, the differences between the two frameworks should be more due to bidders' characteristics than due to a seller's decision, throughout the dissertation I modelled these differences as part of the auction's design. The bidding window rule (and also the bidding fee and the price increment) should be set by seller, when designing the auction.

The case of a sequential bidding auction describes what should happen in a perfect environment (the price moves up immediately after someone submitting a bid). The multiple round bidding auction may seem more controversial, since actual penny auctions do not have a fixed bidding time interval. However, sometimes there can be a lag between the moment when a bidder submits a bid and the moment when the remaining players become aware of that bid. It can happen because bidders are away from each other, sometimes even in different countries, and the internet signal does not move at the speed of light. This lag is specially relevant in short bidding window auctions.

5.1. Choosing the right rule

Although the seller cannot explicitly choose one rule over the other (the multiple round bidding auction rule is never described in bidding fee auctions sites: theoretically, the round should progress immediately after a bid is placed), he can induce bidders' behaviours similar to the ones described in the two models. As I mentioned in Section 2, for very short bidding windows, it is likely that bidders behave as in the multiple round bidding auction, while in auctions with longer windows, bidders may behave more similarly to how it was described in the sequential bidding auction. Then, one can approximate the bidding window rule decision to a bidding window period decision. Additionally, the auctioneer,

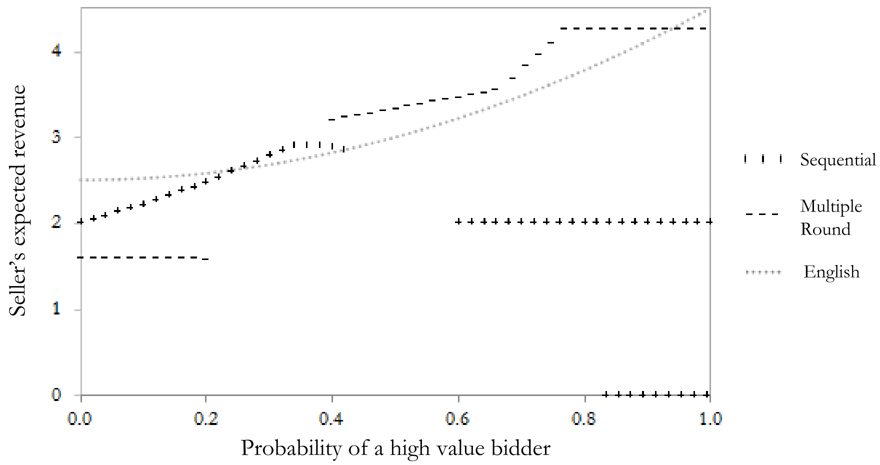

Figure 5: Seller's revenue in the sequential and in the multiple round bidding auction models, comparing with an English auction benchmark.

as the owner of the server that keeps the auction house online, is able to overload or alleviate it. If the server is slow, bidders will not be able to see instantaneous price rises, and will act as in the model of Section 4. On the other hand, if the server is fast, bidders act as described in Section 3. I assume that bidders entering the auction house's website are able to tell which bidding window rule is being used.[33]

Using the parametrized example in Table 2, I check which bidding window rule is more profitable for the seller. Figure 5 overlaps Figure 2 and Figure 4 and it stresses the differences in terms of seller's revenue between the two models. For an initial belief in $[0, \frac{2}{5})$ the seller prefers designing the bidding fee auction with sequential bidding. However, for $[\frac{2}{5}, 1]$, the seller's expected revenue is higher in the multiple round bidding auction.

There is a match between the sequential auction's symmetric equilibrium and the multiple round auction's asymmetric equilibrium revenues in the interval $[0, \frac{1}{3})$. The same is true for $(\frac{3}{5}, 1]$. It is due to the fact that both a multiple round bidding asymmetric equilibrium and a sequential bidding equilibrium allow a unique bid submission in the first period.

If the auctioneer has the possibility of changing the bidding window rule, she

[33] *E.g.*: if there is a short bidding window period and the page is taking too long to load, they assume they are in a multiple round auction.

can profit from it, by choosing the most appropriate rule for a given initial belief. The existence of such a mechanism would also enable at least a (symmetric or asymmetric) perfect Bayesian equilibrium for all initial beliefs, in my example.

5.2. Comparing with the English auction

But should the seller auction off her object using a bidding fee auction mechanism? I also compare the outcome of the bidding fee auctions with the one of a standard English auction. The English auction is a good benchmark since it leads to efficient outcomes. For the same parametrized example, the English auction equilibrium is straightforward: if there are two bidders with the same valuation, one will be the winner, paying her entire willingness to pay ($v_H = 4.5$ or $v_L = 2.5$); if there is a bidder of each type, the v_H will be the winner, paying her rival's valuation, plus an infinitesimal: $v_L + \delta$, with $\delta \to 0^+$. The English auction would provide the seller with an expected revenue of $2.5 + 2\theta_1^2$. This revenue's curve is also represented in Figure 5.

For beliefs in $(\frac{1}{4}, \frac{3}{5})$, the bidding fee auction (with the optimal bidding window rule) is strictly better than the English auction. In its turn, the English auction dominates both bidding fee auction variations for beliefs in the interval $[0, \frac{1}{4})$ and for beliefs greater than $\sqrt{\frac{595}{676}} \approx 0.94$. For other initial expectations, nothing can be said, since it depends on the probability of happening each of the multiple bidding fee auction's equilibria. Assuming each equilibrium has the same probability of occurring, the bidding fee auction would underperform.

The Revenue Equivalence Theorem (Vickrey, 1961) does not hold in bidding fee auctions since (i) the set of possible values is discrete, (ii) it is not true that the bidder with the highest value is always the auction's winner, (iii) it is not true that the bidder with the lowest value expects always zero surplus. Nevertheless it is still interesting that an allocation mechanism based on discrete price increments and in costly bids can sometimes lead to higher profits.

6. Concluding Remarks

This dissertation included two risk-neutral approaches to bidding fee auctions with independent private values. One where the first bidder submitting a bid in a given round was the only one supporting the bidding fee, becoming on the lead in the subsequent round; and another, where each round's successful bidder was the last submitting a bid. The two models are not incompatible.

Rather, they complement each other: both strive to explain what happens in real penny auctions, but in different circumstances.

The extremely high unpredictability of these auctions' returns, described in the media (McCarthy, 2011) and in other working papers on the topic (Hinnosaar, 2010) motivated me to analyze a model of private values and check if bidding fee auctions could present multiple equilibria. My dissertation's conclusion is that they are likely to have it. By generating multiple equilibria, these auctions generate different expected revenues: the models I developed led to multiple perfect Bayesian equilibria with quite different outcomes (at least for a specific family of parameters).

Both the sequential bidding auction and the multiple round bidding auction models may lead to inefficient allocation, either because the object is not sold (and there are bidders with positive willingnesses to pay) or because the object is indeed sold, but the winning bidder is not the one who most values it.

Also, the seller's revenue is not independent of the bidding window rule: the auctioneer can profit from changing the rule according to the beliefs about the buyers' types. In the parametrization equilibria I constructed, there is a region, where the probability of high valuation bidders is lower, for which the seller gains from using the sequential bidding auction rule. On the other hand, if the probability of high valuation bidders is higher, the seller should use the multiple round bidding auction rule.

I also compared bidding fee auctions with the English auction. Even though I had mixed results, in my example, bidding fee auctions could outperform the English auction.

It is not my dissertation's aim to make general conclusions on bidding fee auctions, besides that my findings are possible to happen. I use a parsimonious model of only two bidders, where all agents (bidders and seller) are risk neutral and my analysis is focused only on symmetric and on a specific type of asymmetric equilibria. If I had checked for the existence of other types of equilibria, some of my conclusions could have changed. Another limitation of my model is that even though I allow different bidders' valuations, by setting the seller's reserve utility to zero, I do not let bidders to value the good less than the auctioneer. I also assumed discrete valuations, rather than continuous, which are more likely to happen in reality.

This topic is far from being closed. Further studies should also include a private values analysis with more than two bidders (potentially assuming leader passivity) and it would also be interesting to introduce Byers et al. (2010)'s asymmetries into the model.

Appendix A. Sequential Bidding Auction

Appendix A.1. Timer

In this section I prove that introducing a timer in the sequential bidding auction model is irrelevant.

Definition 7. Let round t be a time interval between $[t, t+1)$.

Consider the case in which more than a person could be willing to submit a bid. In this framework, the successful bidder in each round would be the fastest submitting her bid.

Let $t \leq \varsigma_t^i < t+1$ be the action of bidder i, regarding the bidding moment in the t^{th} bidding decision.

Proposition 4. *All bidders willing to bid try to submit a bid at instant* $\varsigma_t^i = t$.

Proof. I prove that there is no profitable deviation. Consider the problem of a bidder who is willing to bid in round t.

Her utility in the following round is given by:

$$u_{t+1}^i = \begin{cases} u_{t+1}^i(l) - c & if\ b_t^j = 1 \wedge \varsigma_t^i < \varsigma_t^j \vee b_t^j = 0 \\ \frac{1}{2}(u_{t+1}^i(l) - c) + \frac{1}{2}u_{t+1}^i(f) & if\ b_t^j = 1 \wedge \varsigma_t^i = \varsigma_t^j \\ u_{t+1}^i(f) & if\ b_t^j = 1 \wedge \varsigma_t^i > \varsigma_t^j \end{cases}$$

It is impossible to bid at $\varsigma_t^i < t$. A possible deviation must be $\varsigma_t^i > t$, which would give an expected utility of $\rho_t^j \cdot u_{t+1}^i(f) + (1-\rho_t^j)(u_{t+1}^i(l) - c) \leq \rho_t^j(\frac{1}{2}(u_{t+1}^i(l) - c) + \frac{1}{2}u_{t+1}^i(f)) + (1-\rho_t^j)(u_{t+1}^i(l) - c)$, once $u_{t+1}^i(l) - c \geq u_{t+1}^i(f)$ (the bidder is willing to bid). □

In my model, I do not take into consideration aggressive bidding strategies nor other types of strategies that could, for instance, use the bidding timing to signal private values.

Appendix A.2. Obtaining the equilibrium actions

This appendix gives a more detailed explanation about how the equilibria in Section 3 were found.

In any round t bidder i will choose p_t^i that solves the following programme:

$$
\begin{aligned}
\max_{\rho_t^i} \Bigg\{ V_t^i = \rho_t^i \Bigg[\theta_t^j \Bigg(& \rho_t^j(v_H, s_t^j)\Big(\frac{1}{2}(u_{t+1}^i(l) - c) + \frac{1}{2}u_{t+1}^i(f)\Big) \\
& + (1 - \rho_t^j(v_H, s_t^j))\Big(u_{t+1}^i(l) - c\Big)\Bigg) \\
+ (1 - \theta_t^j)\Bigg(& \rho_t^j(v_L, s_t^j)\Big(\frac{1}{2}(u_{t+1}^i(l) - c) + \frac{1}{2}u_{t+1}^i(f)\Big) \\
& + (1 - \rho_t^j(v_L, s_t^j))\Big(u_{t+1}^i(l) - c\Big)\Bigg)\Bigg] \\
+ (1 - \rho_t^i)\Bigg[\theta_t^j\Bigg(& \rho_t^j(v_H, s_t^j) \cdot u_{t+1}^i(f) + (1 - \rho_t^j(v_H, s_t^j)) \cdot ue_t^i(s_t^i)\Bigg) \\
+ (1 - \theta_t^j)\Bigg(& \rho_t^j(v_L, s_t^j) \cdot u_{t+1}^i(f) + (1 - \rho_t^j(v_L, s_t^j)) \cdot ue_t^i(s_t^i)\Bigg)\Bigg]\Bigg\} \qquad \text{(A.1)}
\end{aligned}
$$

$$
s.t.\, \theta_{t+1}^j(\rho_t^j(v^j, s_t^j), s_{t+1}^j)\, \forall v^j \in \{v_H, v_L\},\, \forall s_{t+1}^j \in \{f, l\}
$$

where the ending utility ue_t^i is equal to $v^i - (t-1)\varepsilon$ if bidder i is the leader at t or 0 if she is the follower, θ is the belief system, $s_t^j = s_t^{-i}$ and $u_{t+1}^i(s_{t+1}^i) = u(v^i, s_{t+1}^i, \theta_{t+1}^j)$.

The perfect Bayesian equilibria of the game are achieved by solving this programme for all possible nodes. We can substitute the constraint equations directly into the objective function (in u_{t+1}^i, which depend on the equilibrium actions at $t+1$, which depend on θ_{t+1}^j) and compute its first order condition. Since the programme is linear in ρ_t^i, the *F.O.C.* will lead to either corner solutions or solutions on the whole edge.

F.O.C:

$$
\frac{dV_t^i}{d\rho_t^i} \gtreqless 0 \Leftrightarrow
$$

$$\Leftrightarrow \theta_t^j \Bigg(\rho_t^j(v_H, s_t^j) \Big(\frac{1}{2}(u_{t+1}^i(v^i, l) - c) + \frac{1}{2} u_{t+1}^i(v^i, f) \Big) + (1 - \rho_t^j(v_H, s_t^j)) \Big(u_{t+1}^i(l) - c \Big) \Bigg)$$

$$+ (1 - \theta_t^j) \Bigg(\rho_t^j(v_L, s_t^j) \Big(\frac{1}{2}(u_{t+1}^i(l) - c) + \frac{1}{2} u_{t+1}^i(v^i, f) \Big) + (1 - \rho_t^j(v_L, s_t^j)) \Big(u_{t+1}^i(l) - c \Big) \Bigg) \Bigg]$$

$$- \theta_t^j \Bigg(\rho_t^j(v_H, s_t^j) \cdot u_{t+1}^i(f) + (1 - \rho_t^j(v_H, s_t^j)) \cdot ue_t^i(s_t^i) \Bigg)$$

$$-(1 - \theta_t^j) \Bigg(\rho_t^j(v_L, s_t^j) \cdot u_{t+1}^i(f) + (1 - \rho_t^j(v_L, s_t^j)) \cdot ue_t^i(s_t^i) \Bigg) \gtreqless 0$$

Leading to:
$\rho_t^i = 1$ if $\frac{dV_t^i}{d\rho_t^i} > 0$, $\rho_t^i = 0$ if $\frac{dV_t^i}{d\rho_t^i} < 0$ and $\rho_t^i \in [0, 1]$ if $\frac{dV_t^i}{d\rho_t^i} = 0$.

Appendix A.3. Equilibria of the numerical example

At $t = 4$, a v_L is out of the game and a v_H will wait for sure ($\overline{T} = 4$), so we move back to $t = 3$.

At $t = 3$, $u_4(v_H, l) = 4.5 - 3 - 1 = 0.5$ and $u_4(v_H, f) = 0$. Substituting these values in (A.1), computing its first order condition and solving for θ_3^j, leads to the following bidding conditions and pay-offs:

$$\rho_3^H(f, s_{t-1}^i) = 1 \, \forall \theta_3^j(l, s_{t-1}^j)$$

$$\rho_3^H(l, s_{t-1}^i) = \begin{cases} 1 & if \ \theta_3^j(f, s_{t-1}^j) \in (\frac{8}{9}, 1] \\ [0, 1] & if \ \theta_3^j(f, s_{t-1}^j) = \frac{8}{9} \\ 0 & if \ \theta_3^j(f, s_{t-1}^j) \in [0, \frac{8}{9}) \end{cases}$$

$$u_3(v_H, f, s_{t-1}^i) = 0.5 - 0.25 \rho_3^H(l, s_{t-1}^i) \cdot \theta_3^j(l, s_{t-1}^j)$$

$$u_3(v_H, l, s_{t-1}^i) = \begin{cases} 0.5 - 0.25 \theta_3^j(f, s_{t-1}^j) & if \ \theta_3^j(f, s_{t-1}^j) \in (\frac{8}{9}, 1] \\ 2.5 - 2.5 \theta_3^j(f, s_{t-1}^j) & if \ \theta_3^j(f, s_{t-1}^j) \in [0, \frac{8}{9}] \end{cases}$$

This means that if, by chance, bidders reach $t = 3$, they may face one of four possible situations, depending on the values of $\theta_3^j(f, f)$ and $\theta_3^j(f, l)$ (Table A.9).

	$\theta_3^j(f,l)$	$\theta_3^j(f,f)$	$u_3^H(f,f)$	$u_3^H(f,l)$	$u_3^H(l,f)$	$u_3^H(l,l)$
3.1	$[0,\frac{8}{9})$	$[0,\frac{8}{9})$	0.5	0.5	$2.5-2.5\theta_3^j(f,l)$	$2.5-2.5\theta_3^j(f,l)$
3.2	$[0,\frac{8}{9})$	$(\frac{8}{9},1]$	$0.5-0.25\theta_3^j(l,l)$	0.5	$2.5-2.5\theta_3^j(f,l)$	$0.5-0.25\theta_3^j(f,f)$
3.3	$(\frac{8}{9},1]$	$[0,\frac{8}{9})$	0.5	$0.5-0.25\theta_3^j(l,f)$	$0.5-0.25\theta_3^j(f,l)$	$2.5-2.5\theta_3^j(f,f)$
3.4	$(\frac{8}{9},1]$	$(\frac{8}{9},1]$	$0.5-0.25\theta_3^j(l,l)$	$0.5-0.25\theta_3^j(l,f)$	$0.5-0.25\theta_3^j(f,l)$	$0.5-0.25\theta_3^j(f,f)$

Table A.9: Equilibrium pay-offs at $t=3$, depending on the leader's beliefs about the follower.

	$\theta_2^j(l)$	$\theta_2^j(f)$	$\rho_2^H(f)$	$\rho_2^H(l)$	$u_2^H(f)$	$u_2^H(l)$
2.1	$[0, \frac{3}{5})$	$[0, \frac{16}{17})$	1	0	$1.5 - 2.5\theta_2^j(l)$	$3.5 - 3\theta_2^j(f)$
2.2	$(\frac{3}{5}, \frac{8}{9})$	$[0, \frac{8}{9})$	0	0	0	3.5
2.3	$[0, \frac{3}{5})$	$(\frac{16}{17}, 1]$	1	0	$1.5 - 2.5\theta_2^j(l)$	$3.5 - 3\theta_2^j(f)$
2.4	$(\frac{3}{5}, \frac{8}{9})$	$(\frac{8}{9}, 1]$	0	0	0	3.5
2.5	$(\frac{8}{9}, 1]$	$[0, \frac{8}{9})$	0	0	0	3.5
2.6	$(\frac{8}{9}, 1]$	$(\frac{8}{9}, 1]$	0	0	0	3.5

Note: 2.1 and 2.2 lead to 3.1; 2.3 and 2.4 lead to 3.2; 2.5 leads to 3.3 and 2.5 leads to 3.4

Table A.10: Equilibrium actions and pay-offs at $t = 3$, depending on the leader's beliefs about the follower.

Moving back to $t = 2$, the equilibrium actions depend not only on the value of $\theta_2^j(s_2^j)$ but also on the expectation of reaching each of the four situations in Table A.9.

Replacing the utilities of each case in (A.1) and solving the programme subject to (2) and (3) evaluated at $t = 3$, leads to the equilibrium situations at $t = 2$, in Table A.10.

At $t = 1$, besides v_H bidders, v_L may also be willing to bid. We solve the programme for both types of bidders, considering the four situations that can occur at $t = 2$, for a v_H. Then we check for the existence of symmetric equilibria, and we get the equilibrium strategy profiles in Table 3.

In what concerns the existence of asymmetric equilibria in this framework, it can be claimed that no equilibrium is possible.

Lemma 1. *It is necessary and sufficient for an asymmetric equilibrium to exist, that* $u_2^i(v^i, l) - u_2^i(v^i, f) < c < u_2(l)$, $\forall v^i \in \{v_L, v_H\}$.

Proof. Consider the programme in Appendix A.2 and a candidate equilibrium with $\rho_1^i = 1$ and $\rho_1^j = 0$.

For the one who is bidding not willing to deviate, the expected utility of bidding must be higher than the expected utility of deviating (i.e. waiting): $u_2^i(f) - c > 0$.

For the one who is waiting, the inverse should be true, leading to $u_2^i(f) > u_2^i(l) - c$.

These two inequalities together, lead to the condition above. □

Proposition 5. *There is no asymmetric equilibrium of the model, for the parametrized example in Section 3.2.*

Proof. Consider $u_2^H(l)$ and $u_2^H(f)$ in Table A.10.

Lemma 1 is never verified for a v_H. □

Appendix A.4. Seller's revenue

The seller's revenue is equal to the winning bid plus the fees of all bids that were placed during the auction. Let t^f be the final round of the auction (the first round in which $b_t^i = 0$, $\forall i$). The seller's revenue is:

$$SR(t^f) = (t^f - 1)(c + \varepsilon)$$

The seller's expected revenue considers all possible equilibrium t^f:

$$E[SR] = \sum_{t=1}^{\overline{T}} \Pr(b_t^i = 0, \forall i) \cdot (t-1)(c+\varepsilon)$$

Appendix A.4.1. Seller's revenue in the example

Table A.11 shows what is the seller's revenue in each of the six possible equilibria.

Appendix B. Multiple Round Bidding Auction

Appendix B.1. Timer

Definition 8. Let round t be a time interval between $(t-1, t]$.

In this framework, the successful bidder would be the last-one bidding in a given round.

Let $t - 1 < \varsigma_t^i \leq t$ be the action of bidder i, regarding the bidding moment in the t^{th} bidding decision.

Proposition 6. *All bidders willing to bid try to submit a bid at instant $\varsigma_t^i = t$.*

Proof. I prove that there is no profitable deviation. Consider the problem of a bidder who is willing to bid in round t.

Her utility in the following round is given by:

	θ_1	Seller's revenue
$a)$	$[0, \frac{1}{3})$	$2(1+\theta_1+\theta_1^2)$
$b)$	$[\frac{1}{3}, \frac{3}{7})$	$2(8\theta_1 - 11\theta_1^2)$
$c)$	$(\frac{3}{5}, \frac{8}{9})$	2
$d)$	$(\frac{5}{6}, \frac{16}{17})$	0
$e)$	$(\frac{8}{9}, 1]$	2
$f)$	$(\frac{16}{17}, 1]$	0

Table A.11: Seller's revenue in the parametrized example, assuming a sequential bidding auction.

$$u^i_{t+1} = \begin{cases} u^i_{t+1}(l, p_t + 2\varepsilon) - c & if\ b^j_t = 1 \wedge \varsigma^i_t > \varsigma^j_t \\ \frac{1}{2}(u^i_{t+1}(l, p_t + 2\varepsilon) - c) + \frac{1}{2}u^i_{t+1}(f, p_t + 2\varepsilon) & if\ b^j_t = 1 \wedge \varsigma^i_t = \varsigma^j_t \\ u^i_{t+1}(f, p_t + 2\varepsilon) & if\ b^j_t = 1 \wedge \varsigma^i_t < \varsigma^j_t \\ u^i_{t+1}(l, p_t + \varepsilon) - c & if\ b^j_t = 0 \end{cases}$$

It is impossible to bid at $\varsigma^i_t > t$. A possible deviation must be $\varsigma^i_t < t$, which would give an expected utility of $\rho^j_t \cdot u^i_{t+1}(f, p_t+2\varepsilon) + (1-\rho^j_t)(u^i_{t+1}(l, p_t+\varepsilon) - c) \leq \rho^j_t(\frac{1}{2}(u^i_{t+1}(l, p_t+2\varepsilon) - c) + \frac{1}{2}u^i_{t+1}(f, p_t+2\varepsilon)) + (1-\rho^j_t)(u^i_{t+1}(l, p_t+\varepsilon) - c)$, since $u^i_{t+1}(l, p_t + 2\varepsilon) - c \geq u^i_{t+1}(f, p_t + 2\varepsilon)$ (the bidder is willing to bid). □

Appendix B.2. Obtaining the optimal actions

In each round, a bidder chooses ρ^i_t to maximize her expected utility, given her system of beliefs:

$$\max_{\rho_t^i} \Bigg\{ V_t^i = \rho_t^i \Bigg[\mu_t^j \Bigg(\rho_t^H \Big(\frac{1}{2} u_{t+1}^i (l, p_t + 2\varepsilon) + \frac{1}{2} u_{t+1}^i (f, p_t + 2\varepsilon) - c \Big)$$

$$+ (1 - \rho_t^H) \Big(u_{t+1}^i (l, p_t + \varepsilon) - c \Big) \Bigg)$$

$$+ (1 - \mu_t^j) \Bigg(\rho_t^L \Big(\frac{1}{2} u_{t+1}^i (l, p_t + 2\varepsilon) + \frac{1}{2} u_{t+1}^i (f, p_t + 2\varepsilon) - c \Big)$$

$$+ (1 - \rho_t^L) \Big(u_{t+1} (l, p_t + \varepsilon) - c \Big) \Bigg) \Bigg]$$

$$(1 - \rho_t^i) \Bigg[\mu_t^j \Bigg(\rho_t^H \cdot u_{t+1}^i (f, p_t + \varepsilon) + (1 - \rho_t^H) u e_t^i (s_t) \Bigg)$$

$$+ (1 - \mu_t) \Bigg(\rho_t^L \cdot u_{t+1}^i (f, p_t + \varepsilon) + (1 - \rho_t^L) u e_t^i (s_t) \Bigg) \Bigg] \tag{B.1}$$

$$s.t.\ \mu_t^j (\beta_{t-1}^j (v^j)) \forall v^j \in \{v_H, v_L\}$$

F.O.C:

$$\frac{dV_t^i}{d\rho_t^i} \gtreqless 0 \Leftrightarrow$$

$$\Leftrightarrow \rho_t^H \Big(\frac{1}{2} u_{t+1}^i (l, p_t + 2\varepsilon) + \frac{1}{2} u_{t+1}^i (f, p_t + 2\varepsilon) - c \Big) +$$

$$+ (1 - \rho_t^H) \Big(u_{t+1}^i (l, p_t + \varepsilon) - c \Big) \Bigg) +$$

$$+ (1 - \mu_t^j) \Bigg(\rho_t^L \Big(\frac{1}{2} u_{t+1}^i (l, p_t + 2\varepsilon) + \frac{1}{2} u_{t+1}^i (f, p_t + 2\varepsilon) - c \Big) +$$

$$+ (1 - \rho_t^L) \Big(u_{t+1} (l, p_t + \varepsilon) - c \Big) \Bigg) \Bigg] +$$

$$- \mu_t^j \Bigg(\rho_t^H \cdot u_{t+1}^i (f, p_t + \varepsilon) + (1 - \rho_t^H) u e_t^i (s_t) \Bigg) +$$

$$- (1 - \mu_t) \Bigg(\rho_t^L \cdot u_{t+1}^i (f, p_t + \varepsilon) + (1 - \rho_t^L) u e_t^i (s_t) \Bigg) \gtreqless 0$$

Appendix B.3. Equilibria of the numerical example

At $t = 4$, a v_L is out of the game and a v_H will wait for sure ($\overline{T} = 4$), so we move back to $t = 3$.

p_3	$\mu_3^l(\beta_2^l)$	$\mu_3^f(\beta_2^f)$	$\rho_3^H(f, p_3)$	$\rho_3^H(l, p_3)$	$u_3^H(f, p_3)$	$u_3^H(l, p_3)$
2	$[0, 1]$	$[0, 1]$	1	0	0.5	$2.5 - 2.5\mu_3^f$
3	$[0, 1]$	$[0, 1]$	0	0	0	1.5
4	$[0, 1]$	$[0, 1]$	0	0	0	0.5

Table B.12: Equilibrium actions and pay-offs for $t = 3$.

p_2	$\mu_2^l(b_1^l)$	$\mu_2^f(b_1^f)$	$\rho_2^H(f, p_3)$	$\rho_2^H(l, p_3)$	$u_2^H(f, p_2)$	$u_2^H(l, p_2)$
1	$[0, \frac{3}{5})$	$[0, 1]$	1	0	$1.5 - 2.5\mu_2^l$	$2.5 - 2.5\mu_2^f$
1	$(\frac{3}{5}, 1]$	$[0, 1]$	0	0	0	1.5
2	$[0, 1]$	$[0, 1]$	1	0	0.5	$2.5 - 2.5\mu_2^f$

Table B.13: Equilibrium actions and pay-offs for $t = 2$.

At $t = 3$, $u_4(v_H, l) = 4.5 - p_3 - 1 = 3.5 - p_3$ and $u_4(v_H, f) = 0$. Moreover, at $t = 3$, p_3 can take different values: $p_3 = 2$ if a bid alone was placed in each of the previous two rounds, $p_3 = 3$ if in one of the two preceding rounds both bidders did bid and $p_3 = 4$ if there were two bids in both rounds. Note that p_3 cannot be lower than 2, for $t = 3$ to be reached (otherwise, it would mean that in one preceding round no bid was placed and the game would have ended there) and, since there are only two bidders, p_3 cannot be higher than 4.

Definition 9. $\beta_t^{s_t^i}$ is the vector of past bidding actions of the bidder whose state at t is s_t^i.

Definition 10. $\mu_t^{s_t^j}(\beta_t^{s_t^j})$ is the belief of bidder i about bidder j's type, if bidder j's state at t is s_t^j.

Each bidder's strategy must have an action for round $t = 3$, for all possible p_3. We solve the programme in Appendix B.2 for all possible prices and for the two states. Results are in Table B.12.

Moving back to $t = 2$, $p_2 \in \{1, 2\}$, depending on how many bids there were in $t = 1$. We solve again B.1 for a v_H leader and follower, considering the utility values in Table B.12 to found the equilibrium actions of this period. Results are in Table B.13.

At $t = 1$, v_L bidders might also bid. Depending on μ_1, seven different equilibria are possible (Tables 5 and 6).

Appendix B.4. Seller's revenue

Let t^f be the final round of the auction (the first round in which $b_t^i = 0$, $\forall i$). In this model, the revenue for the seller is given by:

$$SR(t^f) = (c+\varepsilon)\sum_{i=1}^{2}\sum_{t=1}^{t^f-1} b_t^i$$

The seller's expected revenue is:

$$E[SR] = \sum_{t=1}^{\overline{T}}\Big(\Pr(b_t^i = 0\forall i)\cdot(c+\varepsilon)\sum_{i=1}^{2}\sum_{k=1}^{t-1} b_k^i\Big)$$

Appendix B.4.1. Seller's revenue in the example

Table B.14 shows what is the seller's revenue in each of the six possible equilibria.

	μ_1	Seller's revenue
$a)$	$[0, \frac{3}{14}]$	$\frac{2(4-\mu_1^2)}{5}$
$b)$	$[\frac{2}{5}, \frac{2}{3}]$	$\frac{4\mu_1+8}{3}$
$c)$	$(\frac{2}{3}, \frac{10}{13})$	$2\mu_1(2+\mu_1)$
$d)$	$[\frac{10}{13}, 1]$	$\frac{720}{169}$
$e)$	$(\frac{5}{6}, 1]$	0
$f)$	$[0, \frac{1}{3})$	$2(1+\mu_1+\mu_1^2)$
$g)$	$(\frac{3}{5}, 1]$	2

Table B.14: Seller's revenue in the parametrized example, assuming a multiple round bidding auction.

References

Augenblick, N., Jan 2011. Consumer and producer behavior in the market for penny auctions: A theoretical and empirical analysis.

Byers, J. W., Mitzenmacher, M., Zervas, G., Apr 2010. Information asymmetries in pay-per-bid auctions.

Economist, Aug 2009. Mind the pennies.

Gibbons, R., 1992. A primer in game theory. FT Prentice Hall.

Hinnosaar, T., Jan 2010. Penny auctions are unpredictable.

McCarthy, J., 2 Jun 2011. Penny auctions promise savings, overlook downsides. USA Today.
URL `http://www.usatoday.com/tech/news/2011-02-06-penny-auctions_N.htm`

Mittal, S., Mar 2010. Equilibrium analysis of generalized penny auctions.

Platt, B. C., Price, J., Tappen, H., Mar 2012. Pay-to-bid auctions.

Riley, J. G., Samuelson, W. F., Jun 1981. Optimal auctions. The American Economic Review 71 (3), 381–392.

Vickrey, W., 1961. Counterspeculation, auctions and competitive sealed tenders. Journal of Finance (16), 8–37.

Washington Attorney General, sep 2010. Pennybiddr agrees to cash out and refund consumers under agreement with washington attorney general.
URL `http://www.atg.wa.gov/pressrelease.aspx?id=26506`