



## Sir Pinski Rides Again

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**Abstract.** The iterative procedure of removing “almost everything” from a triangle ultimately leading to the Sierpinski’s gasket  $\mathcal{S}$  is well-known. But what is in fact left when almost everything has been taken out? Using the Sir Pinski’s game described by Schroeder [4], we identify two dual sets of invariant points in this exquisite game, and from these we identify points left over in Sierpinski gasket. Our discussion also shows that the chaos game does not generate the Sierpinski gasket. It generates an approximation or, at most, a subset of  $\mathcal{S}$ .

**Keywords:** Sierpinski gasket, Sierpinski points, fractals, Sir Pinski game, chaos game, self-similarity, periodicity.

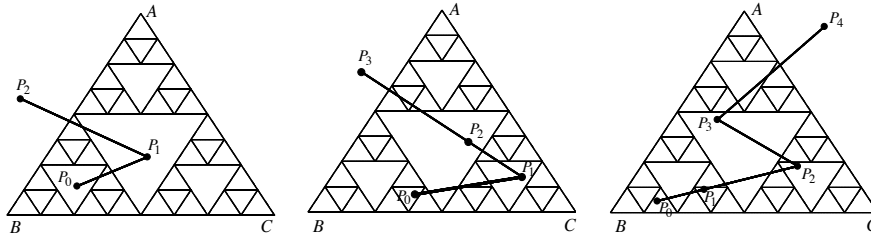
### 1 Introduction

Let  $\mathcal{T}$  be a triangle. A player chooses a point  $P_0$  inside the triangle. Sir Pinski game consists of iteratively jumping to the points  $\{P_1, P_2, \dots\}$ , where  $P_{k+1}$  doubles the distance of  $P_k$  to its nearest vertex. The player looses at step  $n$  if  $P_0, P_1, P_2, \dots, P_{n-1} \in \mathcal{T}$  and  $P_n \notin \mathcal{T}$ .

Let  $\mathcal{M}_1$  denote the “middle triangle” whose vertices are the middle points of the  $\mathcal{T}$  sides. It is obvious that  $\mathcal{M}_1$  is the set of loosing points at step 1. Similarly, the union  $\mathcal{M}_2$  of the three middle triangles of  $\mathcal{T} - \mathcal{M}_1$  is the set of loosing points at step 2, the union  $\mathcal{M}_3$  of the  $3^2$  middle triangles of  $(\mathcal{T} - \mathcal{M}_1) - \mathcal{M}_2 = \mathcal{T} - (\mathcal{M}_1 \cup \mathcal{M}_2)$  is the set of loosing points at step 3, and so on. Loosing points are illustrated in Figure 1, that also clarifies the connection of loosing points at step  $n$  with middle triangles removed at step  $n$  in the classical iterative construction of the Sierpinski gasket.

Schroeder [4] characterizes Sierpinski’s gasket as the set of winning points  $\mathcal{S} = \mathcal{T} - \bigcup_{k=1}^{\infty} \mathcal{M}_k$  of Sir Pinski game.

In fact, from Banach’s contractive mapping fixed point theorem it follows that the Sierpinski gasket  $\mathcal{S} = \mathcal{T} - \bigcup_{k=1}^{\infty} \mathcal{M}_k = \bigcup_{i=1}^3 \psi_i(\mathcal{S})$ , where  $\psi_i(\cdot)$  is the dilation of ratio 1/2 around the vertex  $v_i$  of  $\mathcal{T}$ . In other words,  $\mathcal{S}$  is the unique non-empty fixed point of the corresponding Hutchinson operator [2]  $\psi$ , where  $\psi(\mathcal{A}) = \psi_1(\mathcal{A}) \cup \psi_2(\mathcal{A}) \cup \psi_3(\mathcal{A})$ , i.e.  $\psi(\mathcal{A}) = \mathcal{A}$  if and only if  $\mathcal{A} = \mathcal{S}$ .



**Fig. 1.** Loosing points at steps 2 (*left*), 3 (*center*) and 4 (*right*).

Hence the Sierpinski points  $s \in \mathcal{S}$  can be characterized as the set of points  $s \in \mathcal{T}$  such that  $(s + v_i)/2 \in \mathcal{S}$ ,  $i = 1, 2, 3$ . So, starting from whatever point  $P \in \mathcal{T}$ , iteratively jumping for a point halving the distance to  $v_i$  creates an infinite sequence of points in a straight line that ultimately converges to  $v_i$ . Observe however that

- if  $P \in \mathcal{S}$ , all the iterates are Sierpinski points; but, on the other hand,
- if  $P \notin \mathcal{S}$ , none of the iterates is a Sierpinski point.

In fact, the halving contractions  $\psi_i$  generate points that are nearer and nearer to Sierpinski points, but as the Sir Pinski game clearly shows their inverse doubling ultimately leaves  $\mathcal{T}$  unless the starting point is itself a Sierpinski point.

Iteratively halving (or, alternatively, doubling) the distance to a fixed vertex  $v_i$  creates an infinite sequence of colinear points. Hence we need some rule to use in turn, either deterministically or randomly, the different vertices in order to approximate the Sierpinski gasket  $\mathcal{S}$ . Sir Pinski game uses the rule: take the nearest vertex to the starting point/iterate, and double the distance. A “dual” rule is: use the farthest vertex from the starting point/iterate, and halve the distance (randomly choose one of the vertices when there is a tie) — this rule has the advantage of always changing the vertex to use in sequential steps of the algorithm.

Barnsley [1] devised a *chaos game*, using randomness to generate the three sets  $\psi_i(\mathcal{T})$ : pick a starting point  $P_0$ , and generate iterates  $\{P_1, P_2, \dots\}$ , such that  $P_k$  is the midpoint of the segment whose endpoints are  $P_{k-1}$  and one of the vertices  $v_i$  of  $\mathcal{T}$ , randomly chosen using the discrete uniform law  $X = \begin{Bmatrix} v_1 & v_2 & v_3 \\ 1/3 & 1/3 & 1/3 \end{Bmatrix}$ . This chaos game is generally presented as a device to generate the Sierpinski gasket  $\mathcal{S}$ , but in view of the above observations it produces in general an approximation of the Sierpinski gasket, since in general  $P_0 \notin \mathcal{S}$ . Observe also that even starting from a Sierpinski point, what we obtain is a subset of the Sierpinski gasket — for example, the choice of the top vertex of the equilateral triangle used in [3], page 306, will generate as iterates only vertex points of the triangles left out when middle triangles are removed, in the classical deterministic iterative construction of  $\mathcal{S}$ . This issue will be discussed later in further detail.

## 2 The points of the Sierpinski gasket

As seen in the introduction, the points  $s \in \mathcal{S}$  are easily described using the concept of self-similarity and its far-reaching consequences.

Using translation and rotation, if needed, we assume that the vertices of  $\mathcal{T}$  are  $v_L = (0, 0)$ ,  $v_R = (a, 0)$ ,  $a > 0$ , and  $v_T = (c, d)$ ,  $d > 0$ .

If  $\mathcal{T}$  is the triangle with vertices  $v_L = (0, 0)$ ,  $v_R = (1, 0)$ , and  $v_T = (0, 1)$ , in dyadic notation its Sierpinski points are  $s = (x, 1 - x)$ , i.e. if the abscissa is  $x = 0.\nu_1\nu_2\nu_3\dots$ , the  $k$ -th digit of the ordinate is  $1 - \nu_k$  — for instance,  $s = (0.11001011101\dots, 0.00110100010\dots)$ , cf. Peitgen *et al.* [3], p. 173.

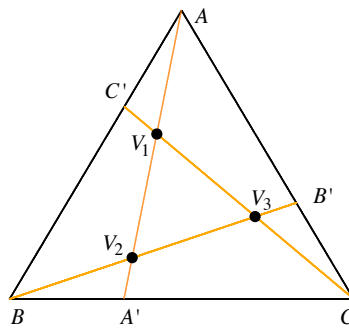
Let  $\mathcal{T}$  be the equilateral triangle with unit height, vertices  $v_L = (0, 0)$ ,  $v_R = (2\sqrt{3}/3, 0)$ , and  $v_T = (\sqrt{3}/3, 1)$ . Schroeder [4], pp. 22–24, used a sophisticated redundant three-coordinates points affixation to show that the Sierpinski points are those with coordinates (in dyadic expansion)  $x = 0.a_1a_2a_3\dots$ ,  $y = 0.b_1b_2b_3\dots$ ,  $z = 0.c_1c_2c_3\dots$ , such that  $(a_k, b_k, c_k) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ ,  $k = 1, 2, \dots$

Let us now consider that  $\mathcal{T}$  is the equilateral triangle with unit sides, with top vertex  $A = (1/2, \sqrt{3}/2)$ , left vertex  $B = (0, 0)$ , and right vertex  $C = (1, 0)$ . Project  $A$  in the point  $A' = (1/3, 0)$ ,  $B$  in  $B' = (5/6, \sqrt{3}/6)$ , and  $C$  in  $C' = (1/3, \sqrt{3}/3)$ .

We claim that the points

- $V_1 = (3/7, 2\sqrt{3}/7)$ , intersection of  $\overline{AA'}$  with  $\overline{CC'}$ ,
- $V_2 = (5/14, \sqrt{3}/14)$ , intersection of  $\overline{AA'}$  with  $\overline{BB'}$ , and
- $V_3 = (5/7, \sqrt{3}/7)$ , intersection of  $\overline{BB'}$  with  $\overline{CC'}$ ,

are Sierpinski points, cf. Figure 2.

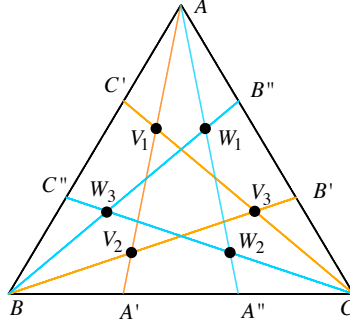


**Fig. 2.** Period-3 invariant Sir Pinski  $\{V_1, V_2, V_3\}$  attractor.

In fact,  $V_1$  is the midpoint of  $\overline{AV_2}$ ,  $V_2$  is the midpoint of  $\overline{BV_3}$ ,  $V_3$  is the midpoint of  $\overline{CV_1}$ , and therefore those points are winning points in the Sir Pinski game, i.e.  $\{V_1, V_2, V_3\}$  is an invariant cycle-3 attractor of Sierpinski points.



Project  $A$  in the point  $A'' = (2/3, 0)$ ,  $B$  in  $B'' = (2/3, \sqrt{3}/3)$ , and  $C$  in  $C'' = (1/6, \sqrt{3}/6)$ . Obviously, intersecting  $\overline{AA''}$  with  $\overline{BB''}$  we obtain  $W_1 = (4/7, 2\sqrt{3}/7)$ , intersecting  $\overline{AA''}$  with  $\overline{CC''}$  we obtain  $W_2 = (9/14, \sqrt{3}/14)$ , and intersecting  $\overline{BB''}$  with  $\overline{CC''}$  we obtain  $W_3 = (2/7, \sqrt{3}/7)$ . For similar reasons,  $\{W_1, W_2, W_3\}$  is an invariant cycle-3 attractor of Sierpinski points, cf. Figure 3.



**Fig. 3.** Period-3  $\{V_1, V_2, V_3\}$  and  $\{W_1, W_2, W_3\}$  invariant Sir Pinski points attractors.  $\{A\}$ ,  $\{B\}$  and  $\{C\}$  are invariant in Sir Pinski game;  $\{A', A''\}$ ,  $\{B', B''\}$  and  $\{C', C''\}$  are period-2 invariant sets in Sir Pinski game.

*Remark 1.* If we re-scale multiplying by  $2/\sqrt{3}$  in order to have unit heights (i.e., each vertex is at distance 1 from the opposite side), the ordinates of the transformed  $V_1^*$  and  $W_1^*$  become  $4/7$ , the ordinates of the transformed  $V_2^*$  and  $W_2^*$  become  $1/7$ , and the ordinates of the transformed  $V_3^*$  and  $W_3^*$  become  $2/7$ .

Hence, if we adhere to Schroeder [4] three-coordinates system  $(x, y, z)$ , where  $x$  is the distance from the bottom side,  $y$  the distance from the left side, and  $z$  the distance from the right side, we see that the period-3 invariant points must have  $x$ -coordinate  $4/7, 1/7$  or  $2/7$ .

From the  $(2\pi/3)$ -rotational symmetry of  $\mathcal{T}$ , it follows that in Schroeder's three coordinates system  $V_1^* = (4/7, 1/7, 2/7)$ ,  $V_2^* = (1/7, 2/7, 4/7)$ ,  $V_3^* = (2/7, 4/7, 1/7)$ ,  $W_1^* = (4/7, 2/7, 1/7)$ ,  $W_2^* = (1/7, 4/7, 2/7)$ , and  $W_3^* = (2/7, 1/7, 4/7)$ .  $\square$

*Remark 2.* The points  $V_1, V_2, V_3, W_1, W_2, W_3$  lie on a circumference of radius  $\sqrt{21}/21$  centered at the barycenter  $(1/2, \sqrt{3}/6)$  of  $\mathcal{T}$ .  $\square$

*Remark 3.* Each vertex of  $\mathcal{T}$  is invariant in Sir Pinski game. Hence  $A, B, C \in \mathcal{S}$ . On the other hand, in Sir Pinski game, the image of  $A'$  is  $A''$  and vice-versa, i.e.  $\{A', A''\}$  is a period-2 invariant set, and the same holds for  $\{B', B''\}$



and  $\{C', C''\}$ .  $\mathcal{V} = \{V_1, V_2, V_3\}$  and  $\mathcal{W} = \{W_1, W_2, W_3\}$  are period-3 invariant sets (attractors) in Sir Pinski game.

Higher order periodic invariant sets do exist. For instance, using conditions  $(a - 1/2)^2 + (b - \sqrt{3}/2)^2 = 4[(2a - 1/2)^2 + (2b - \sqrt{3}/2)^2]$  and  $(2b - \sqrt{3}/2)/(2a - 1/2) = (\sqrt{3}/2 - b)/(a - 1/2)$  on the set of points  $\{(a, b), (2a, 2b), (1 - a, b), (1 - 2a, 2b)\}$ , so that  $(a, b) = (0.3, 0.288675)$ , we obtain the period-4 invariant set  $\{(0.3, 0.288675), (0.6, 0.636194), (0.7, 0.288675), (0.4, 0.636194)\}$ , cf. Figure 4.

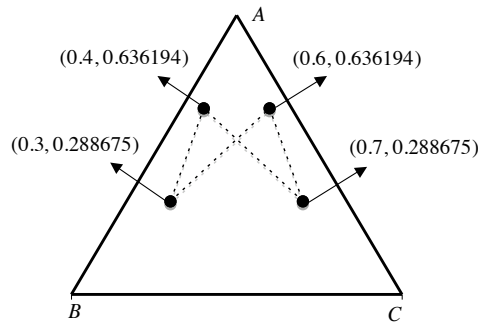


Fig. 4. A period-4 invariant Sir Pinski set.

Using the  $(2\pi/3)$ -rotational symmetry of  $\mathcal{T}$ , two other period-4 invariant sets are readily obtained.  $\square$

Now we perform the same construction in the  $T_1$  (Top),  $L_1$  (Left) and  $R_1$  (Right) triangles remaining once the middle triangle of  $\mathcal{T}$  is removed in step 1 of the classical construction of the Sierpinski gasket, obtaining  $2 \times 3^2$  points —  $3^2$   $V$ s and  $3^2$   $W$ s —, as shown in Figure 5. With the

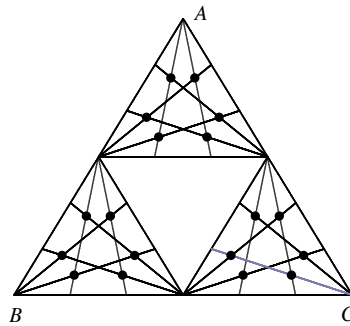


Fig. 5. More Sierpinski points, in  $T_1$ , in  $L_1$  and in  $R_1$ .

self-explaining addressing and notations  $V_{i,L_1}, W_{i,L_1}$ ,  $i = 1, 2, 3$ , it is obvious

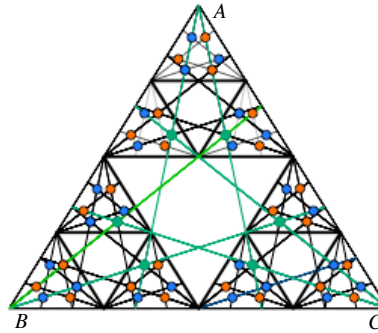


that  $V_{i,L_1} = \frac{1}{2} V_i$  and  $W_{i,L_1} = \frac{1}{2} W_i$  — for instance,  $V_{2,L_1} = (5/28, \sqrt{3}/28)$ ,  $V_{1,L_1} = (4/14, 2\sqrt{3}/14)$ .

Analogously, the corresponding points in the Right triangle  $R_1$  are  $V_{i,R_1} = (1/2, 0) + 1/2 V_i$  and  $W_{i,R_1} = (1/2, 0) + 1/2 W_i$ , and the corresponding points in the Top triangle  $T_1$  are  $V_{i,T_1} = (1/4, \sqrt{3}/4) + 1/2 V_i$  and  $W_{i,T_1} = (1/4, \sqrt{3}/4) + 1/2 W_i$ . For instance,  $V_{1,T_1} = (13/28, 11\sqrt{3}/28)$ .

The  $3^2$   $V$  points in this second stage of the construction are, following the above algorithm,  $(3/14, \sqrt{3}/7)$ ,  $(5/7, \sqrt{3}/7)$ ,  $(13/28, 11\sqrt{3}/28)$ ,  $(5/28, \sqrt{3}/28)$ ,  $(19/28, \sqrt{3}/28)$ ,  $(3/7, 2\sqrt{3}/7)$ ,  $(5/14, \sqrt{3}/14)$ ,  $(6/7, \sqrt{3}/14)$ ,  $(1/28, 9\sqrt{3}/28)$  — exactly the 9 points we obtain when we compute the middle point of the segments joining each of the  $(3/7, 2\sqrt{3}/7)$ ,  $(5/14, \sqrt{3}/14)$ ,  $(5/7, \sqrt{3}/7)$   $V$  points from stage one of the construction with each of the three vertices of  $\mathcal{T}$ . Similar results hold in what concerns  $W$  points.

Continuing the procedure, in step 3 of the iterative construction of Sierpinski's gasket we obtain  $2 \times 3^3$  points as shown in Fig. 6. (We have included some extra segments connecting points to make clear that in Sierpinski game whatever the initial  $V$  point [respectively,  $W$  point], in a few steps we shall land in the attractor  $\mathcal{V} = \{V_1, V_2, V_3\}$  [respectively, in  $\mathcal{W} = \{W_1, W_2, W_3\}$ ].)



**Fig. 6.** More Sierpinski points, in  $T_1$ , in  $L_1$  and in  $R_1$ .

Once again the coordinates of any  $V$  or  $W$  point are easy to compute. For instance  $W_{1,L_1T_2} = (1/8, \sqrt{3}/8) + (1/2)^2 \times (4/7, 2\sqrt{3}/7) = (9/56, 11\sqrt{3}/56)$ , since the left vertex of the triangle whose address is  $L_1T_2$  is  $(1/8, \sqrt{3}/8)$ .

Using the same line of reasoning, the  $V_{i,R_1T_2R_3}$  points of  $R_1T_2R_3$  will have coordinates  $(3/4, \sqrt{3}/8) + (1/2)^3 V_i$ , the  $W_{i,R_1L_2T_3T_4}$  points of  $R_1L_2T_3T_4$  will have coordinates  $(13/16, (13/16)(\sqrt{3}/2)) + (1/2)^4 W_i$ . More generally,

- in step  $n$ , the coordinates of the original  $V$ s and  $W$ s are scaled by a factor  $(1/2)^n$ ;
- the address determines the left vertex of the triangle: a  $L_k$  does not affect neither the abscissa nor the ordinate, a  $R_k$  shifts the left corner  $(1/2)^k$



and does not affect the ordinate, and a  $T_k$  adds  $(1/4)^k$  to the abscissa and  $(1/2)^k \sqrt{3}/2$  to the ordinate.

For instance, the left corner of  $T_1L_2L_3R_4R_5T_6$  is  $(1/4 + (1/2)^4 + (1/2)^5 + (1/4)^6, (1/2 + (1/2)^6)(\sqrt{3}/2)) = (1409/4096, 33\sqrt{3}/128)$ . Hence, the Sierpinski point  $W_{3,T_1L_2L_3R_4R_5T_6}$  is  $(1409/4096, 33\sqrt{3}/128) + (1/2)^6(2/7, \sqrt{3}/7) = (10119/28672, 233\sqrt{3}/896)$ .

*Remark 4.* Suppose that in the  $k$ -th step of the iterative deterministic construction of the Sierpinski gasket we focus our attention in one of the remaining triangles, for instance  $T_1R_2R_3T_4 \cdots L_k$ .

- The midpoints of the segments whose endpoints are the vertex  $A$  and the points of  $T_1R_2R_3T_4 \cdots L_k$  are the points of  $T_1T_2R_3R_4T_5 \cdots L_{k+1}$ .
- The midpoints of the segments whose endpoints are the vertex  $B$  and the points of  $T_1R_2R_3T_4 \cdots L_k$  are the points of  $L_1T_2R_3R_4T_5 \cdots L_{k+1}$ .
- The midpoints of the segments whose endpoints are the vertex  $C$  and the points of  $T_1R_2R_3T_4 \cdots L_k$  are the points of  $R_1T_2R_3R_4T_5 \cdots L_{k+1}$ .

Hence, the chaos game transforms the  $V$  points [respectively, the  $W$  points] of  $T_1R_2R_3T_4 \cdots L_k$  in  $V$  points [respectively,  $W$  points] of either  $T_1T_2R_3R_4T_5 \cdots L_{k+1}$ , or  $L_1T_2R_3R_4T_5 \cdots L_{k+1}$  or  $R_1T_2R_3R_4T_5 \cdots L_{k+1}$ .  $\square$

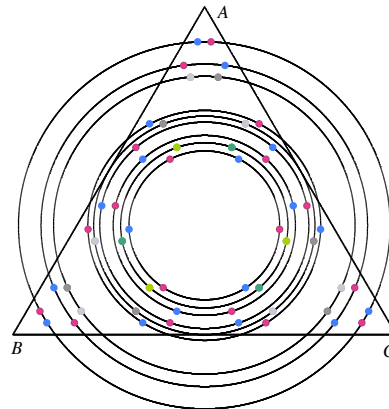
It seems useless to elaborate more on this matter to conclude that:

- In the  $k$ -th step of the classical construction of the Sierpinski gasket we may explicitly compute the coordinates of 3  $V$  points and of 3  $W$  points in each remaining triangle.
- The midpoint of any  $V$  point [respectively,  $W$  point] and any vertex of  $\mathcal{T}$  is a  $V$  point [respectively, a  $W$  point]. In other words, in the chaos game the set of  $V$  points and the set of  $W$  points do not communicate.
- In Sir Pinski game, a  $V$  starting point generates iterates that ultimately will land in  $\mathcal{V}$ , and a  $W$  starting point generates iterates that ultimately will land in  $\mathcal{W}$ . Hence all  $V$  and  $W$  points are winning points of the Sir Pinski game, i.e. they lie in  $\mathcal{S}$ . We say that  $V$  points [respectively,  $W$  points] are in the attraction domain of  $\mathcal{V}$  [respectively, of  $\mathcal{W}$ ], or that  $\mathcal{V}$  and  $\mathcal{W}$  are invariant periodicity-3 attractors in Sir Pinski game.

*Remark 5.* We also observe that subsets of 3  $V$  points and 3  $W$  points lie in circumferences centered at the barycenter of  $\mathcal{T}$ , cf. Fig. 7.  $\square$

### 3 Concluding Remarks

Under the heading “Randomness Creates Deterministic Shapes”, Peitgen *et al.* [3], p. 299, raise some interesting questions. The discussion in the previous section patently shows that the chaos game does not generate the Sierpinski gasket.



**Fig. 7.** A consequence of the  $\frac{2\pi}{3}$ -rotational symmetry of  $\mathcal{S}$

More precisely, if the starting point  $P_0$  is not a Sierpinski point, its descendants are not Sierpinski points, and eventually some of them computed in the initial steps are clearly spurious specks observed upon close scrutiny of the images. The set looks like the Sierpinski gasket, because the composition of contractions creates something that is very close to the Sierpinski gasket, but its intersection with the Sierpinski gasket  $\mathcal{S}$  is void.

On the other hand, our discussion shows that sets generated by the chaos game starting with a Sierpinski  $V$  point and with a Sierpinski  $W$  point are disjoint. Moreover, any of them leaves out points in the domain of attraction of invariant attractors with periodicities other than 3.

So, even with a carefully selected Sierpinski point in any of those invariant sets, the best we can get applying the chaos game is a rarefied pale image of the rich complexity of the Sierpinski gasket. The gross imperfection of the representation of points and our eyes trick us in believing we are generating the Sierpinski gasket. In fact, the representation we get is as inaccurate as the representation we get after a finite number of steps of removal of middle triangles, in the classical deterministic construction.

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