Profit Reversal in Symmetric Ownership Structures: A Study of Cournot and Bertrand Competition

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Dissertation written under the supervision of Professor Duarte Brito.

Dissertation submitted in partial fulfilment of requirements for the MSc in Economics, at the Universidade Católica Portuguesa, January 2025.

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Abstract

This paper examines the impact of passive symmetric ownership on market out- comes in Cournot and Bertrand competition within a vertically structured model. The model includes one upstream firm that supplies the same input to two down- stream firms, and focuses on the conditions that lead to profit reversal between the two competition types. Symmetric ownership is analyzed in three configurations: Horizontal Cross-Ownership (HCO), Partial Vertical Forward Ownership (PVFO), and Partial Vertical Backward Ownership (PVBO). The findings reveal that profit reversal at the downstream level only occurs under PVBO, and depends on the ownership share and the degree of product differentiation. Additionally, the study examines profit reversal at both the upstream and industry levels across all configu- rations. The analysis also explores the impact of ownership structures on consumer welfare and market efficiency, providing strategic insights for firms and practical guidance for regulatory policies in vertically integrated industries.

Keywords: Cournot, Bertrand, Profit Reversal, Symmetric Ownership

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Resumo

Este artigo examina o impacto da propriedade simétrica passiva nos resultados de mercado de concorrência à la Cournot e à la Bertrand, no contexto de uma indústria estruturada verticalmente. O modelo inclui uma empresa a montante (upstream) que fornece um fator pro- dutivo a duas empresas a jusante (downstream), concentrando-se nas condições que levam à reversão dos lucros nos dois tipos de concorrência. A propriedade simétrica é analisada em três configurações: Propriedade Cruzada Horizontal (HCO), Propriedade Vertical Parcial das empresas a Jusante pela empresa a Montante (PVFO) e Propriedade Vertical Parcial da empresa a Montante pelas empresas a Jusante (PVBO). Os resultados demonstram que a reversão de lucros a jusante ocorre exclusivamente no caso de PVBO e depende do nível de propriedade e do grau de difer- enciação do produto. Adicionalmente, o estudo examina a possibilidade de reversão de lucros a montante e ao nível da indústria em todos os tipos de propriedade. A análise explora ainda o impacto das estruturas de propriedade no bem-estar do consumidor e na eficiência do mercado, oferecendo informações estratégicas para empresas e orientações práticas para políticas regulatórias em indústrias verticalmente integradas.

Palavras-chave: Cournot, Bertrand, Profit Reversal, Symmetric Ownership

Extended Abstract

This paper examines the impact of passive symmetric ownership, on market outcomes in Cournot and Bertrand competition within a vertically structured model. The model includes one upstream firm that supplies the same input to two downstream firms, and focuses on the conditions that lead to profit reversal between the two competition types. Symmetric ownership is analyzed in three configurations: Horizontal Cross-Ownership (HCO), Partial Vertical Forward Ownership (PVFO), and Partial Vertical Backward Own- ership (PVBO).

The findings reveal that profit reversal at the downstream level occurs exclusively under PVBO, and depends on the ownership share and the degree of product differenti- ation. At the upstream level, profits are always higher under Bertrand competition for HCO and PVFO, whereas in PVBO, upstream profit reversal depends on the parameters. However, industry profits are always higher at Bertrand in the three ownership structures, regardless of parameter values.

The analysis further explores how ownership structures affect consumer welfare and market efficiency, both of which are found to be higher under Bertrand competition in all cases. By employing a less complex ownership structure than Mukherjee et al (2024), this thesis identifies the fundamental causes of profit reversal in vertically integrated industries with symmetric ownership. These results provide strategic insights for firms and practical guidance for regulatory policies in vertically integrated industries.

Acknowledgements

First and foremost, I wish to express gratitude to my thesis supervisor, Professor Duarte Brito. His guidance, patience, and expertise have been indispensable to my work. His insightful advice and thoughtful suggestions provided a strong foundation for the successful completion of this thesis.

I am eternally thankful to my parents, Célia Cantinho and Paulo Silva, whose unconditional love, encouragement, and steadfast support have been my anchor throughout this journey.

Finally, I owe immense thanks to Francisco Santos for his incredible support, motiva- tion, and unyielding patience during the most challenging moments. His faith in me and willingness to stand by my side gave me the strength to persevere.

Contents

1 Introduction

Traditionally, Cournot competition is associated with higher firm profitability due to less aggressive competitive pressures, when compared to Bertrand competition. However, under certain market structures, this result is challenged, and conventional economic theory is up-turned. This thesis explores cases of profit reversal, examining the conditions under which Bertrand competition proves more profitable than Cournot competition.

This research investigates profit reversal within symmetric vertical structures with- out control, a framework that reflects realistic industrial settings, while simplifying the ownership arrangements often studied in the literature. Building on the work of Mukher- jee, Wang, and Sun (2024), who demonstrated profit reversal at downstream level within an elaborate cross-ownership framework, this study examines whether simpler ownership configurations can yield similar outcomes and explores the mechanisms driving these out- comes.

The Mukherjee et al (2024) model involves a vertically integrated industry with one upstream supplier and two downstream firms, where each firm holds symmetric shares in all the others. Their findings show that the profits generated in the final good pro- ducers are higher under Bertrand competition when cross-ownership levels and product differentiation are high.

While groundbreaking, their model relies on a complex network of ownership rela- tions, prompting the question: Are such elaborate structures necessary to observe profit reversal, or can simpler ownership configurations also produce these effects? To address this, three streamlined ownership configurations, commonly found in vertically integrated industries, are explored: Horizontal Cross-Ownership (HCO), Partial Vertical Forward Ownership (PVFO), and Partial Vertical Backward Ownership (PVBO). In HCO, down- stream firms hold shares in their competitors, softening competition by internalizing rival profits. In PVFO, the upstream supplier owns shares in the downstream firms, influencing downstream behavior through upstream profit alignment. In PVBO, downstream firms hold shares in the upstream supplier, which reduces double marginalization and fosters closer integration. These structures are prevalent in industries such as telecommunica- tions, energy, and manufacturing, underscoring the practical relevance of this study.

Using a two-stage game-theoretic framework, this thesis analyzes how these ownership structures interact with competition, shaping market outcomes. The findings indicate that Bertrand competition always outperforms Cournot in terms of industry profits, consumer surplus, and social welfare, across all configurations.

However, the dynamics vary: In HCO and PVFO, downstream profits are higher un- der Cournot competition, while upstream profits are higher under Bertrand competition. In PVBO, profit reversal at downstream level becomes feasible, particularly when own- ership shares are high and product differentiation is high (more independent products). However, the upstream profits under Bertrand competition surpass those under Cournot when ownership shares are low, and product differentiation is low (more substitutable products).

According to Singh and Vives (1984), Cournot competition is typically more profitable for downstream firms with imperfect substitutes. However, this thesis shows that under PVBO, in certain conditions, Bertrand competition can be more profitable for final good producers, aligning with Mukherjee et al (2024) findings.

By simplifying the ownership framework, this thesis not only extends the findings of Mukherjee et al (2024) but also provides deeper insights into the mechanisms driving profit reversal. The findings underscore the importance of parameter interactions and provide strategic insights for firms, in vertically integrated industries, considering owner- ship strategies, and aid policymakers to foster competitive and efficient markets. They are particularly relevant in contexts where firms can choose their strategic variable, whether price or quantity.

The thesis is structured as follows: the next section reviews the relevant literature. Section 3 presents the model and its key assumptions. Section 4 solves the model for the three distinct ownership configurations, comparing Cournot and Bertrand competition with a focus on profits, consumer surplus, and social welfare. Section 5 concludes, and all proofs are presented in the appendix.

2 Literature Review

This study examines the effects of symmetric ownership without control on market outcomes, in Cournot and Bertrand competition, within a vertically structured model, bearing resemblance to literature that compares quantity and price competition, and also with cross-ownerships.

One of the main concerns in industrial organization theory is the comparison between Bertrand and Cournot competition, where Singh and Vives (1984) provide foundational analysis. The authors study the firm's decisions with respect to their strategic variables (prices or quantities), in a differentiated duopoly, when the goods can either be substitutes or complements. They found that firms have a dominant (profit) strategy to choose quantity competition (Cournot) for substitute goods, and price competition (Bertrand) for complementary goods. However, in a differentiated duopoly Bertrand competition is always superior to Cournot for consumer surplus and total welfare, as it leads to lower prices and higher quantities.

Singh and Vives' work is critical for the following research made on competition outcomes, in varying market structures, but does not address cross-ownership.

With respect to horizontal cross-ownership, Flath (1991) examines its impact on competition outcomes, in a purely horizontal model, with n firms. Flath finds that acquiring shares in rivals tends to soften competition raising prices and profits, in Bertrand competition. In contrast, in Cournot competition, share acquisitions reduce competitive intensity but do not improve operational profits.

Although the common view is that horizontal cross-ownership reduces competition and welfare, there are exceptions. In a similar model, Vives (2020) considers the positive effects of technological spill-overs that arise from common and cross-ownership.[1] Although these ownership structures make Bertrand competition more profitable by internalizing rivals' profits and softening competition, Vives finds that common ownership also boosts R&D investment, leading to higher welfare through increased innovation.

In many industries, however, vertical structures are common, with upstream suppli- ers providing essential inputs to downstream firms that compete to sell final goods. The simplest case (with the minimum number of firms), in a vertical structure, involves one up- stream supplier and two competing downstream firms. Other variations include multiple upstream suppliers, additional downstream firms, or vertically integrated firms competing at the downstream level.

Some related articles discuss the Cournot-Bertrand profit differential in vertical indus- tries, but without considering cross-ownership.

This is the case of Correa-López and Naylor (2004), who consider a model wherein an upstream supplier negotiates the input price with each downstream firm. The standard result (Cournot being more profitable than Bertrand) will change when an upstream firm has an advantageous bargaining position and prioritizes high wages. They find the elasticity of labor demand to be higher at Bertrand competition, weakening the upstream's bargaining power and leading to lower input prices. Therefore, downstream firms face lower input costs and consequently earn higher profits under Bertrand competition.

[1] Common ownership occurs when institutional investors, such as mutual funds, hold shares in multiple competing firms within an industry, typically without direct control. Cross-ownership happens when one firm holds shares in a competing firm.

Analogously to previous work, in Correa-López (2007) competing à la Bertrand be- comes more profitable when the bargaining power of downstream' firms is decentralized (each downstream firm negotiates its own input price), so firms have more bargaining power and can negotiate lower input prices.

Arya et al (2008) demonstrate that profit reversal might occur when input production is outsourced to a vertically integrated firm.[2] The vertically integrated firm has market power and prioritizes its profit maximization, setting a high input price. The input price is even higher under Bertrand, as the upstream firm uses its power to soften the competitive downstream behavior, resulting in higher industry profits, but lower consumer surplus and social welfare.

Vertical structures become more complex when cross-ownership is involved, where firms across different levels of the supply chain hold ownership stakes in each other. This can be seen in industries like telecommunications, energy, and manufacturing.

The effects of cross-ownership without control in vertical structures, under Cournot and Bertrand competition, have been discussed by many authors.[3]

Luciano Fanti (2016) adds a vertical dimension to examine the impact of asymmet- ric horizontal cross-ownership, where each downstream firm has an exclusive upstream supplier. Under horizontal cross-ownership social welfare is higher under Bertrand com- petition as the input prices will be lower, comparatively to Cournot. This is an in- teresting result, where although downstream firms compete less aggressively, due to the cross-ownership, the reduction in input price outweighs the negative effect on competition, leading to higher social welfare.

In a similar framework as Fanti (2016), without upstream exclusivity, Hunold (2013) – (Kapitel 2) analyzes passive backward ownership, where downstream firms acquire stakes in upstream suppliers, without control. The analysis considers that under Bertrand com- petition, ownership stakes can lead to price increases, amplifying the effects of ownership on reducing competition, and enforcing that antitrust effects are felt strongly with partial ownership structures.

Our paper departs from these articles in assuming that there is a common input

[2] A vertically integrated firm is one that operates both upstream, producing an essential input, and downstream, competing in the retail market.

[3] For an example of related work where there is control see Levy, Spiegel, and Gilo (2018) who found that, in a vertical structure (with two downstream and two upstream firms), ownership can lead to foreclosure. They identify two scenarios where this occurs: when a downstream firm (D1) partially integrates by acquiring a stake in an upstream supplier (U1), and when U1 acquires a stake in D1. These integrations do not require full control but enough to influence the strategic decision of the firms. This provides an interesting insight, particularly in the context of antitrust policy.

supplier, instead of exclusive input suppliers. The two papers that are most closely related to ours (using the same vertical industry framework) are Greenlee & Raskovich (2006) and Mukherjee, Wang, and Sun (2024).

Greenlee & Raskovich (2006) explore the impact of PVBO on market competition. In symmetric Cournot competition with homogeneous goods, they find that PVBO can have minimal impact on total output. However, in asymmetric cases under Cournot, PVBO can amplify cost asymmetries, shifting output toward lower-cost firms and improving total surplus but potentially harming consumer welfare due to higher prices. In contrast, under Bertrand competition, PVBO has more adverse effects on consumer welfare in differenti- ated goods markets. Bertrand settings often lead to higher prices, as firms with ownership stakes benefit from raising their rivals' costs. The model does not allow the comparison of Cournot and Bertrand results, because the former is derived for homogeneous products and the latter for differentiated products.

Mukherjee, Wang, and Sun (2024) found that, in a two-tier industry with simultaneous cross-ownership among all firms, downstream profits can be higher under Bertrand com- petition than under Cournot competition, particularly for high degrees of cross-ownership (α) and product differentiation (r). Their model assumes passive cross-ownership (α), with $\alpha \in [0; \frac{1}{3}]$, an assumption that is consistent with the framework developed in this thesis.

A key contribution of their work is the graphical analysis, which identifies a specific region, for particular values of α and r, where both upstream and downstream profits are higher under Bertrand competition compared to Cournot competition. This region is crucial because it represents the only scenario in which Bertrand competition domi- nates Cournot across all dimensions: upstream profits, downstream profits, total industry profits, consumer surplus, and social welfare. Outside this region, although industry prof- its, consumer surplus, and welfare are always higher under Bertrand competition, profit advantages may not hold simultaneously for both the upstream and downstream firms.

However, in their model, the simultaneous presence of HCO, PVFO, and PVBO, make it challenging to disentangle the root cause of the profit reversal. For instance, the interac- tion between PVFO and PVBO can produce conflicting effects on input prices and profit distribution, depending on the values of parameters. These complexities underscore the importance of isolating individual ownership structures to better understand the drivers of profit reversal.

Building on these findings, this thesis builds on their findings by examining simplified and distinct configurations of cross-ownership, aiming to better isolate and understand the mechanisms driving profit reversal.

3 Theoretical Framework and Model Setup

The model examines a vertical industry structure, with an upstream monopolist, firm U, supplying an essential input to two symmetric downstream firms, i and j. The model considers three distinct symmetric ownership structures, which introduce financial links between firms at different levels of the supply chain or at the same level. The share each firm holds in another is represented by α. The parameter $\alpha \in \left(0; \frac{1}{2}\right)$, reflects partial ownership without control, which is standard in models of passive ownership structures.

The three scenarios are as follows:

1. Horizontal Cross-Ownership (HCO): Each downstream firm owns a share α of the other downstream firm.
2. Partial Vertical Forward Ownership (PVFO): The upstream monopolist owns a share α of each downstream firm.
3. Partial Vertical Backward Ownership (PVBO): Each downstream firm holds a share α of the upstream monopolist.

The analysis is built upon several key assumptions:

First, the model assumes symmetry between the downstream firms i and j in terms of production costs and market power, ensuring no firm holds an inherent competitive advantage. This assumption of symmetry allows the model to focus specifically on the effects of partial ownership on competition outcomes, without the added complexity of firm-level asymmetries.

A second assumption is that partial ownership does not confer control over opera- tional decisions. In this model, ownership shares do not directly influence firms' strategic decisions, each firm independently maximizes its own profit, which includes any financial interests in other firms. This assumption is standard in theoretical models that focus on the effects of ownership structure while excluding control issues.

Finally, the upstream monopolist sets a uniform input price u (long-term fixed and without price discrimination) and operates with zero production costs.[4]. For simplicity, producing one unit of output at the downstream level requires one unit of input from firm U, so u also represents the marginal cost for downstream firms i and j. Let q_i and q_j denote the quantities of input purchased from firm U by downstream firms i and j,

[4] Symmetry between downstream firms makes input price discrimination an irrelevant issue.

respectively. Each firm's output quantity is derived from these input purchases.

The utility function of a representative consumer for the outputs of firms i and j is given by:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2}$$

This utility function yields the inverse and direct demand function as $P_i = 1 - q_i - rq_j$ and $q_i = \frac{1}{r+1} - \frac{P_i}{1-r^2} + r\frac{P_j}{1-r^2}$ respectively, where $i \neq j$. These demand functions link the output choices of each firm to the resulting market prices (or link the price set by the firms with the corresponding sales), which depend on the degree of substitutability between the products, r. Hence, downstream firms sell imperfect substitutes, where the degree of product differentiation is represented by $r \in [0, 1]$. When $r = 0$, the products are fully independent, indicating maximum differentiation. Conversely, when $r = 1$, the products are perfect substitutes, implying no differentiation.

The operational profits of the downstream firms are, under Cournot competition (expressed as a function of quantities), given by:

$$v_i^C = v_j^C = (1 - q_i - rq_j - u)\, q_i$$

Under Bertrand competition, these profits (expressed as a function of prices) are:

$$v_i^B = v_j^B = (P_i - u)\frac{1}{1 - r^2}(1 - r - P_i + rP_j)$$

The upstream input producer's operational profit is merely given by:

$$v_u = u(q_i + q_j)$$

The industry's operational profit is represented by:

$$v_{IND} = v_u + 2v_i$$

All firms' total profit functions (what firms maximize) include part of their operational profit, and also their (silent) financial participation in the other firms' operational profit. The total profits for the downstream firms are represented by N_i and N_j, while the upstream firm's total profit is represented by N_u.[5]

[5] As expected, the total profits depend on the specific ownership structure under consideration. De- tailed equations for total profits corresponding to each ownership structure are provided in the appendix for reference.

The game has two stages:

Stage 1: The upstream monopolist chooses the optimal input price u to maximize its profit.

Stage 2: Downstream firms i and j simultaneously choose their output quantities q_i and q_j (under Cournot competition) or set prices P_i and P_j (under Bertrand competition), maximizing their profits in each scenario.

This game is solved using backward induction, beginning with the downstream firms' decisions in the second stage and working back to the upstream firm's optimal pricing decision in the first stage.

4 Models: Cases of symmetric partial ownership

This section examines firms' profits, consumer surplus, and social welfare, in three scenarios of symmetric ownership: HCO, PVFO, PVBO. For each case, the outcomes in terms of profits and welfare will be analyzed and compared under both Cournot and Bertrand competition frameworks.

Under HCO, with maximization of downstream joint profits (their own plus the finan- cial interest in their competitor), competition is softened because downstream firms inter- nalize a portion of rival's profits. This results in downstream firms setting higher prices or producing less output, thereby reducing their input' demand. Contrary to downstream firms, the upstream firm's incentives remain unchanged with cross-ownership. Having no costs, the upstream monopolist maximizes its revenue by selecting the input price that ensures the elasticity of demand equals one. As shown below, this results in the same input price under both Bertrand and Cournot competition.

Under PVFO, the downstream firms have the same incentives as without ownership, thus keeping the same competitive behavior and the same input demand. Since the competitive behavior of the downstream firms remains unchanged, the aggregate demand for inputs retains the same functional relationship with the input price, irrespective of the degree of ownership. However, while the input demand function and its elasticity do not depend on ownership, the equilibrium input price does. This is because the upstream firm partially internalizes the downstream firms' profits, incentivizing it to set a lower input price to benefit the downstream firms. This adjustment benefits downstream firms without altering their strategic incentives.

Under PVBO, with maximization of downstream joint profits (their own plus the financial interest in the upstream firm), downstream firms internalize part of the upstream profit. This internalization of a portion of the upstream profit leads downstream firms to react less sensitively to increases in the input price, making the input demand more rigid as α increases. Specifically, the input price elasticity of demand decreases with higher levels of ownership. Despite this, the competitive dynamics between downstream firms remain unaffected by this ownership structure. Furthermore, this type of ownership does not alter upstream incentives, as its decisions remain focused solely on maximizing its operational profits.

By examining these three scenarios, we can identify the distinct mechanisms through which partial ownership shapes market outcomes and influences the relative profitability under Cournot and Bertrand competition. HCO primarily softens downstream competi- tion, while PVFO and PVBO impact upstream pricing strategies and downstream demand sensitivity in different ways. This analysis sets the stage for comparing profits, consumer surplus, and social welfare, under the two competition frameworks.

4.1 Horizontal (Downstream) Cross-Ownership (HCO)

We start by presenting the equilibria both under Cournot and Bertrand competition. A discussion follows.

Lemma 1 (HCO): In the presence of HGO and under Gournot competition:

a) fhe equilibrium input price is

$$u^{C\square} = \frac{1}{2}$$

b) Equilibrium quantities are

$$q_i^{C\square} = q_j^{C\square} = \frac{1}{2} \frac{1-\alpha}{2(1-\alpha)+r}$$

c) Equilibrium prices are

$$p_i^{C\square} = p_j^{C\square} = \frac{r(\alpha+1)+3(1-\alpha)}{2(2(1-\alpha)+r)}$$

d) Equilibrium profits are

$$v_i^{C\square} = v_j^{C\square} = \frac{1}{4}(1-\alpha)\frac{1-\alpha+\alpha r}{(2(1-\alpha)+r)^2}$$

$$v_u^{C\square} = \frac{1-\alpha}{2(1-\alpha)+r}$$

$$v_{IND}^{C\square} = \frac{(2r-5\alpha+\alpha r+5)(1-\alpha)}{2(r-2\alpha+2)^2}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{C\square} = \frac{(r+1)(1-\alpha)^2}{4(2(1-\alpha)+r)^2}$$

$$SW^{C\square} = \frac{(r(\alpha+3)+7(1-\alpha))(1-\alpha)}{4(2(1-\alpha)+r)^2}$$

As expected, the equilibrium quantity decreases with the degree of cross-ownership, due to less competition between downstream firms. This decrease is even more pronounced when products are closer substitutes. With cross-ownership, downstream firms internalize part of the loss they impose on their competitor by increasing output, leading them to reduce their quantities.

Cross-ownership, at the downstream level, does not affect the input price. This con- stancy arises from the assumption of a linear demand function. As demonstrated in the Proof of Lemma 1 (HCO), where $u^{C\square} = \frac{1}{2}$, the demand for the input is given by $Q^C(u) = \frac{2(1-\square)}{2(1-\square)+r}(1-u)$. The linear demand curve makes the term with α (and r) just a multiplicative constant in the profit function, which effectively removes cross-ownership from influencing the upstream pricing decision.

As a result, downstream operational profits $\left(v_i^{C\square}, v_j^{C\square}\right)$ increase with α (a higher degree of cross-ownership pushes the market closer to a monopolistic outcome), while the upstream firm's operational profits $\left(v_u^{C\square}\right)$ decrease due to reduced output. The overall industry profits $\left(v_{IND}^{C\square}\right)$ decrease as α increases, because the double marginalization problem intensifies.[6] Consumer surplus and social welfare also decline as α increases.

We now turn to price competition.

Lemma 2 (HCO): In the presence of HGO and under Bertrand competition:

[6] The double marginalization problem arises from cumulative markups in a vertical structure, where both upstream and downstream firms exercise market power to maximize their individual profits. This sequential pricing leads to inflated final prices, reduced output, and inefficiencies, ultimately harming overall welfare and industry profits.

a) fhe equilibrium input price is

$$u^{B\square} = \frac{1}{2}$$

b) Equilibrium quantities are

$$q_i^{B\square} = q_j^{B\square} = \frac{1}{2} - \frac{1-\alpha-\alpha r}{2(r+1)(2-2\alpha-r)}$$

c) Equilibrium prices are

$$p_i^{B\square} = p_j^{B\square} = \frac{\alpha r - 2r - 3\alpha + 3}{2(2-2\alpha-r)}$$

d) Equilibrium profits are

$$v_i^{B\square} = v_j^{B\square} = \frac{1}{4}\frac{(1-\alpha)(r-1)(\alpha+\alpha r-1)}{(2\alpha+r-2)^2(r+1)}$$

$$v_u^{B\square} = \frac{\alpha+\alpha r-1}{2(2\alpha+r-2)(r+1)}$$

$$v_{IND}^{B\square} = \frac{(1-\alpha-\alpha r)(\alpha r-2r-3\alpha+3)\,2}{(2\alpha+r-2)^2(r+1)}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{B\square} = \frac{(\alpha+\alpha r-1)^2}{4(2\alpha+r-2)^2(r+1)}$$

$$SW^{B\square} = \frac{(1-\alpha-\alpha r)(\alpha r-4r-7\alpha+7)}{4(2\alpha+r-2)^2(r+1)}$$

As expected, as cross-ownership increases, the equilibrium price set by each down- stream firm increases and this effect is stronger when products are closer substitutes (higher r). A higher degree of cross-ownership reflects increasing incentives for down- stream firms to soften competition. If r increases, their products become closer substi- tutes, and therefore it is easier for firms to set higher prices. If r increases, their products become closer substitutes and the demand for product i is more sensitive to the price of product j. When firms care for the other's profits due to cross-holdings, they internalize part of the rivals benefit from a higher price (which increases with r). Thus, a higher r leads to higher prices.

The fact that the input price does not change with α, $u^{B\square} = \frac{1}{2}$, simplifies this relationship by removing any cost variability due to input price changes. (The justification for this is the same as under Cournot competition).

Downstream operational profits v^{B}, Y^{B}_{j} increase with higher cross-ownership because the price increase outweighs the reduction in quantity produced (the price is closer to the monopoly price which would be obtained for $\alpha = 1$), and the input price remains unaffected by cross-ownership.[7] However, the upstream operational profits v^{B}_{u} decrease as cross-ownership increases, due to reduced input production at the same equilibrium input price. Overall, the total industry profits v^{B}_{IND} decrease with α.

Consumer surplus declines as cross-ownership increases, driven by lower quantities and higher prices, that result from reduced competition. Social welfare follows the same pat- tern, as the reduction in consumer surplus coupled with lower upstream profits, outweigh the gains of the downstream firms.

The input price is the same in both Cournot and Bertrand because ownership occurs at the downstream level, and the type of competition does not change the incentives of the upstream firm. The next proposition formalizes this comparison by analyzing the equilibria under the two competitive structures.

Proposition 1 (HCO): Under HGO, input price (u) is the same in Gournot and Bertrand competition ($u^{C} = u^{B}$).

As the level of cross-ownership increases (α), the demand for the input decreases in the two competition types, because the downstream firms sell less output.[8] However, the assumptions on the downstream industry are such (linearity) that these changes do not affect the input demand elasticity ($o^{C} = o^{B} = \frac{m}{1-m}$) and therefore have no impact on the monopolist's input price.

Proposition 2 (HCO): Under HGO the domnstream firms have higher profits mhen they compete on quantities. fhe upstream profits, total industry profits, consumer surplus, and social melfare, are higher mhen firms compete on prices.

In terms of profitability, downstream firms compete less aggressively under HCO, mak-

[7]The monopoly price is $P^{B}_{i} = P^{B}_{j} = \frac{1}{2}$

[8]Indeed,

$$\frac{6^{C}(r)}{6} = \frac{2(r-1)g}{(2-g-2)^2} < 0$$

$$\frac{6^{B}(r)}{6} = -2\frac{(1-g)(1-r)g}{(2+g-2)^2(g+1)} < 0$$

ing Cournot more advantageous for them, Singh & Vives (1984). In Cournot, downstream firms can charge higher prices, which outweigh the loss in quantity sold, resulting in higher profits for downstream firms compared to Bertrand $\left(v^{C} - v^{B} > 0\right)$.

For the upstream firm, however, Bertrand competition proves more profitable ($v^{C}_{u} - v^{B}_{u} < 0$). The more intense price competition among downstream firms in Bertrand results in a higher quantity demand at a lower price. This higher demand, for the same input price ($u^{C} = u^{B}$), allows the upstream firm to produce more output under Bertrand, resulting in greater profitability for the upstream firm compared to Cournot.

When considering total industry profitability, Bertrand competition leads to a superior outcome ($v^{C}_{IND} - v^{B}_{IND} < 0$). Despite downstream firms earning more in Cournot, the upstream firm's higher profits under Bertrand outweigh this advantage, resulting in greater overall industry profitability. This outcome is driven by the intensified price com- petition under Bertrand, which reduces double marginalization and enhances efficiency across the industry.

In summary, under HCO, due to identical input prices, the differences in profitability arise solely from the intensity of competition. Cournot benefits downstream firms due to softened competition from cross-ownership, whereas Bertrand benefits the upstream firm and the entire industry through increased demand. The downstream result indi- cates that, in this case, this ownership structure does not result in the profit reversal outcome identified by Mukherjee et al (2024), confirming that HCO does not drive this phenomenon.

As previously mentioned, downstream firms base their decisions on their total profits, which naturally align with operational profits since both firms' operational profits move in the same direction under Cournot or Bertrand competition. Therefore, downstream firms achieve higher profitability under Cournot competition, even when total profits are considered $\left(N^{C} > N^{B}\right)$.

Consumer surplus is higher in Bertrand, because quantity is higher and prices are lower under Bertrand competition. Furthermore, Bertrand competition provides higher social welfare than Cournot competition because the consumer surplus is higher at Bertrand, as well as the total industry profits.

The findings align with traditional economic insights, Singh & Vives (1984), which demonstrate that, when $\alpha = 0$, Bertrand competition generally provides higher consumer surplus and social welfare compared to Cournot competition in markets with substitutable products (high r).

$$q_i^{C\square} - q_j^{B\square} = \frac{(1-2\alpha)\, r^2}{2\,(2\alpha - r - 2)\,(2 - r - 2\alpha)\,(r+1)} < 0$$

$$P_i^{C\square} - P_j^{B\square} = \frac{(1-2\alpha)\, r^2}{2\,(2\alpha + r - 2)\,(2\alpha - r - 2)} > 0$$

Fanti (2016) also highlights how cross-ownership softens competition by making firms internalize the negative effects of their output or price decisions on competitors. This soft- ening effect is particularly strong in Cournot competition, where higher cross-ownership reduces equilibrium quantities and raises prices.

By contrast, Bertrand competition mitigates these anti-competitive effects. In Fanti's analysis, even with cross-ownership, Bertrand results in higher quantities and lower prices than Cournot. Consequently, Bertrand delivers better outcomes for consumer surplus and social welfare, as it promotes lower prices and greater output.

The fixed input price ($u^{C\square} = u^{B\square} = \frac{1}{2}$) in this model replicates the stabilizing and pro-competitive effects of reduced input prices observed in Fanti's study. By holding the input price constant, the model highlights how Bertrand competition is more favorable to consumers, as it inherently promotes lower prices and higher output downstream. This simplification removes the complexity of endogenous input price adjustments while still showcasing Bertrand's structural advantages for consumers.

4.2 Partial Vertical Forward Ownership (PVFO)

We start by presenting the equilibria both under Cournot and Bertrand competition. A discussion follows:

Lemma 1 (PVFO): In the presence of PVFO and under Gournot competition:

a) fhe equilibrium input price is

$$u^{C\square} = \frac{2 - 2\alpha + r}{4 - 2\alpha + 2r}$$

b) Equilibrium quantities are

$$q_i^{C\square} = q_j^{C\square} = \frac{1}{2\,(2 - \alpha + r)}$$

c) Equilibrium prices are

$$P_i^{C\square} = P_j^{C\square} = \frac{r - 2\alpha + 3}{2\,(r - \alpha + 2)}$$

d) Equilibrium profits are

$$\pi_i^{C*} = \pi_j^{C*} = \frac{1}{4(r-\alpha+2)^2}$$

$$\pi_u^{C*} = \frac{(r-2\alpha+2)}{2(r-\alpha+2)^2}$$

$$\pi_{IND}^{C*} = \frac{r-2\alpha+3}{2(r-\alpha+2)^2}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{C*} = \frac{r+1}{4(\alpha-r-2)^2}$$

$$SW^{C*} = \frac{3r-4\alpha+7}{4(r-\alpha+2)^2}$$

The equilibrium quantity increases with higher cross-ownership, and this is even stronger when the products are more independent (lower r), because of the internalization of the upstream decision.

As expected, the equilibrium input price decreases with α, because when the share that the upstream firm has on the downstream firms is high, the upstream prioritize downstream profits, and chooses a lower input price. This lower input price encourages higher downstream profits, which indirectly benefit the upstream firm through its shares. The reduction in the input price is more significant when the products are closer substitutes (higher r), as this intensifies competition between downstream firms.

As a result, downstream operational profits (π_i^{C*}, π_j^{C*}) increase with α, while the upstream operational profits (π_u^{C*}) decrease due to the lower input price. However, total industry profits (π^{C*}_{IND}) increase with α, as the reduction in double marginalization improves overall efficiency and profitability. Consumer surplus and social welfare increase with α. The lower input price, at higher levels of cross-ownership, leads to more compet- itive downstream pricing and higher output, benefiting consumers.

We now turn to price competition.

Lemma 2 (PVFO): In the presence of PVFO and under Bertrand competition:

a) fhe equilibrium input price is

$$w^{B*} = \frac{1}{2}\,\frac{-2\alpha-r+2\alpha r+2}{-\alpha-r+\alpha r+2}$$

b) Equilibrium quantities are

$$q_i^{B*} = q_j^{B*} = \frac{1}{2\,(r+1)\,(2-\alpha-r+\alpha r)}$$

c) Equilibrium prices are

$$P_i^{B*} = P_j^{B*} = \frac{2\alpha r-2r-2\alpha+3}{2\,(\alpha r-r-\alpha+2)}$$

d) Equilibrium profits are

$$v_i^{B*} = v_j^{B*} = \frac{1-r}{4\,(\alpha r-r-\alpha+2)^2\,(r+1)}$$

$$v_u^{B*} = \frac{2\alpha r-r-2\alpha+2}{2\,(\alpha r-r-\alpha+2)^2\,(r+1)}$$

$$v_{IND}^{B*} = \frac{2\alpha r-2r-2\alpha+3}{2\,(\alpha r-r-\alpha+2)^2\,(r+1)}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{B*} = \frac{1}{(r+1)\,(\alpha r-r-\alpha+2)^2}$$

$$SW^{B*} = \frac{2\alpha r-2r-2\alpha+5}{2\,(\alpha r-r-\alpha+2)^2\,(r+1)}$$

As expected, the equilibrium prices set by downstream firms decrease for higher levels of cross-ownership, and this effect is stronger when products are closer substitutes (higher r).

As α increases, the input price decreases. This occurs because the upstream firm, holding a share in each downstream firm, has an incentive to lower the input price to increase downstream profitability. Additionally, higher substitutability, reflected by an increase in r, also leads to a lower input price. This reduction is due to intensified competition among downstream firms when products are closer substitutes (higher r).

Downstream operational profits $\left(v_i^{B*}, v_j^{B*}\right)$ increase at higher levels of ownership due to the higher output and lower input price. However, the upstream operational profits $\left(v_u^{B*}\right)$ decrease with higher levels of ownership, because the decrease on the input price outweighs the increase in quantity produced.

Overall, total industry profits $\left(v_{IND}^{B*}\right)$ rise with α, while the double marginalization is mitigated, improving industry efficiency. Consumer surplus and social welfare increase with α.

Proposition 1 (PVFO): Under PVFO, input price (u) is higher under Bertrand competition ($u^{B\square} > u^{C\square}$):

$$u^{C\square} - u^{B\square} = -\frac{r^2\alpha}{2(\alpha r - r - \alpha + 2)(r - \alpha + 2)} < 0$$

Increasing u has a negative impact on downstream profits, which is stronger under Cournot competition. As a result, the upstream supplier prefers to set a higher input price under Bertrand competition ($u^{B\square} > u^{C\square}$), where downstream profits are less sensitive to input cost increases.

$$\frac{6v_i^C}{6u} - \frac{6v_i^B}{6u} = -\frac{4r^3(1-u)}{(r+1)(2-r)^2(r+2)^2} < 0$$

This difference increases as α increases, indicated by the negative derivative of $u^{C\square} - u^{B\square}$ with respect to α. As ownership (α) rises, the gap between input prices under Cournot and Bertrand competition intensifies, increasing the incentive for the upstream firm to set distinct input prices between the two models of competition. However if $\alpha = 0$ (no ownership) or if $r = 0$ (two monopolies at downstream level), the input price is equal under the two types of competition ($u^{C\square} = u^{B\square}$).

$$\frac{6\,\square u^{C\square} - u^{B\square}\square}{6\alpha} = -\frac{1}{2}\frac{(\alpha^2 r - r^2 - \alpha^2 + 4)\,r^2}{(\alpha r - r - \alpha + 2)^2(\alpha - r - 2)^2} < 0$$

In summary, the higher input price under Bertrand competition ($u^{B\square} > u^{C\square}$) is driven by the upstream supplier's stronger incentives. These incentives stem from higher down- stream quantities and reduced sensitivity of downstream profits to input costs in Bertrand competition, effects that become even more pronounced as α increases.

Proposition 2 (PVFO): Under PVFO the domnstream firms have higher profits mhen they compete on quantities. fhe upstream profits, total industry profits, consumer surplus, and social melfare, are higher mhen firms compete on prices.

From a profitability perspective, Cournot competition is more advantageous for down- stream firms, resulting in $v_i^{C\square} - v_i^{B\square} > 0$ (and it would be even if the input price was the same).[9] Conversely, Bertrand competition is more profitable for the upstream firm,

[9] This follows from

$$\pi_i^C - \pi_i^B = \frac{(r-1)^2}{(g+2)^2}(1-\square) - \frac{(r-1)^2(g-1)(\square-1)}{(g-2)^2(g+1)} = \frac{2(1-\square)(r-1)^2 g^3}{(g+2)^2(g-2)^2(g+1)} > 0\square$$

$v_u^{C\square} - v_u^{B\square} < 0$, as it can charge a higher input price without significantly reducing the quantity demanded. Due to the lower profit sensitivity to input cost in Bertrand, down- stream firms are able to absorb higher input prices.

Overall, total industry profits are higher under Bertrand competition, $v^{C\square}_{IND} - v^{B\square}_{IND} < 0$, as the upstream firm's increased profits in Bertrand outweigh the additional profits that downstream firms achieve in Cournot.

These results indicate that, in this case, Cournot competition is more advantageous for downstream firms, showing that this ownership structure does not result in the profit reversal outcome identified by Mukherjee et al (2024), and confirming that PVFO does not drive this phenomenon.

As previously mentioned, downstream firms base their decisions on total profits, which are proportional to their operational profits. Therefore, downstream firms achieve higher profitability under Cournot competition, and this result remains qualitatively unchanged when total profits are considered $N_i^{C\square} > N_i^{B\square}$.

Consumer surplus is higher in Bertrand, because quantity is higher and prices are lower under Bertrand competition. This is aligned with the traditional cases, Singh & Vives (1984).

$$q_i^{C\square} - q_j^{B\square} = \frac{(\alpha - 1)\, r^2}{2\,(\alpha - r - 2)\,(\alpha + r - \alpha r - 2)\,(r + 1)} < 0$$

$$P_i^{C\square} - P_j^{B\square} = \frac{(\alpha - 1)\, r^2}{2\,(\alpha r - r - \alpha + 2)\,(\alpha - r - 2)} > 0$$

In Arya et al (2008), when $\alpha_1 = \alpha_2$ (providing a comparison to PVFO through guar- anteed symmetry), the two downstream firms have the same efficiency in their operations, ensuring symmetry in input costs. This prevents foreclosure, allowing the upstream sup- plier to set a uniform input price to maximize its profits. This is consistent with the findings in PVFO, the article shows that, when $\alpha_1 = \alpha_2$, the input price under Bertrand competition is higher than under Cournot ($u^B > u^C$). Similarly reflecting the reduced sensitivity of downstream profits to input costs in Bertrand competition.

At the industry level, Bertrand surpasses Cournot competition because the upstream supplier's additional profit, from higher input prices, outweighs the reduced profits of downstream firms. This leads to higher total industry profits under price competition. Similarly, the case of PVFO confirms that Bertrand competition results in lower down- stream prices due to fiercer competition, enhancing consumer surplus. As a result, social

where $r^{C\square} = r^{B\square} = r$

welfare is also higher under Bertrand competition.

Overall, the findings of PVFO reinforce and extend the insights provided by Arya et al, under symmetric ownership conditions. The alignment between my results and Arya et al's, demonstrates that, in the absence of foreclosure, competitive dynamics are governed solely by input pricing strategies and downstream competition. Bertrand competition delivers higher welfare, lower prices, and a redistribution of profits favoring the upstream supplier.

4.3 Partial Vertical Backward Ownership (PVBO)

We start by presenting the equilibria both under Cournot and Bertrand competition.
A discussion follows.

Lemma 1 (PVBO): In the presence of PVBO and under Gournot competition:

a) fhe equilibrium input price is

$$u^{C\square} = \frac{1}{2(1 - \alpha)}$$

b) Equilibrium quantities are

$$q_i^{C\square} = q_j^{C\square} = \frac{1}{2(r+2)}$$

c) Equilibrium prices are

$$P_i^{C\square} = P_j^{C\square} = \frac{r+3}{2(r+2)}$$

d) Equilibrium profits are

$$v_i^{C\square} = v_j^{C\square} = \frac{1}{4}\frac{3\alpha + \alpha r - 1}{(r+2)^2(\alpha - 1)}$$

$$v_u^{C\square} = \frac{1}{2(1-\alpha)(r+2)}$$

$$v_{IND}^{C\square} = \frac{1}{2}\frac{r+3}{(r+2)^2}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{C□} = \frac{(r + 1)}{4\,(r + 2)^2}$$

$$SW^{C□} = \frac{3r + 7}{4\,(r + 2)^2}$$

The equilibrium quantity is independent of the level of ownership, because it depends only on the downstream decisions and on the effective input price. Moreover, the equi- librium quantity is negatively related to the degree of substitutability (r), when products are less substitutable (low r), each downstream firm produces a higher optimal quantity. This outcome reflects consumers' preference for product differentiation in an oligopolistic setting.

As expected, the input price increases with α, because downstream firms partially internalize the upstream's profits, leading to a more rigid input demand.

Although the input price increases with α, the effective input price faced by each downstream firm is only a fraction (1 — α) of this price. This is because, when a down- stream firm pays the input price, u, to the upstream firm, it contributes to the upstream firm's profits. A portion (α) of these profits is then redistributed back to the downstream firm due to its ownership stake in the upstream firm. Consequently, the downstream firm effectively experiences an input price equivalent to $\frac{1}{2}$, regardless of the value of α. Total downstream profits[10], however, depend on α because each downstream firm receives both the share of profits from their own, and from the rival's payment of the input price to the upstream firm. This redistribution mechanism explains why total downstream profits are affected by α[11], as it introduces a link between the upstream profits generated by both firms' input price payments, and the distribution of these profits back to the downstream firms.

However, this constancy does not hold for downstream operational profits $\left(v_i^{C□}, v_j^{C□}\right)$. Since the input price increases with the degree of ownership, downstream operational profits decrease with α, while upstream operational profits $\left(v_u^{C□}\right)$ increase. Industry profits

10 $$\Pi_i^{C□} = \Pi_j^{C□} = \frac{1}{4}\,\frac{3□ + □g — 1}{(g + 2)^2\,(□ — 1)} + \frac{□}{2\,(1 — □)\,(g + 2)} = \frac{(□ + □g + 1)}{4\,(1 — □)\,(g + 2)^2}$$

11 $$\frac{6\,\Pi_i^{C□}}{6□} = \frac{6\,\Pi_j^{C□}}{6□} = \frac{1}{4\,(□ — 1)^2\,(g + 2)} > 0$$

$\square\, v_{IND}^{C\square}\, \square$ are independent of ownership, the operational downstream profits decrease with α, due to the increase on the input price, which perfectly outweighs the upstream profit increase with α. Consumer surplus and social welfare also do not depend on α.

We now turn to price competition.

Lemma 2 (PVBO): In the presence of PVBO and under Bertrand competition:

a) fhe equilibrium input price is

$$u^{B\square} = \frac{1}{2\,(1 - \alpha\,(1 - r))}$$

b) Equilibrium quantities are

$$q_i^{B\square} = q_j^{B\square} = \frac{1}{2\,(r+1)\,(2 - r)}$$

c) Equilibrium prices are

$$P_i^{B\square} = P_j^{B\square} = \frac{3 - 2r}{2\,(2 - r)}$$

d) Equilibrium profits are

$$v_i^{B\square} = v_j^{B\square} = \frac{1}{4}(1 - r)\,\frac{-3\alpha + 2\alpha r + 1}{(r+1)\,(r-2)^2\,(-\alpha + \alpha r + 1)}$$

$$v_u^{B\square} = \frac{-1}{2\,(r+1)\,(-\alpha + \alpha r + 1)\,(r - 2)}$$

$$v_{IND}^{B\square} = \frac{3 - 2r}{2\,(r-2)^2\,(r+1)}$$

e) Equilibrium consumer surplus and social melfare are

$$CS^{B\square} = \frac{1}{4\,(r+1)\,(r-2)^2}$$

$$SW^{B\square} = \frac{7 - 4r}{4\,(r-2)^2\,(r+1)}$$

The downstream equilibrium price does not depend on the level of ownership (α), and it is higher for products that are more independent (lower r).

Meanwhile, the input price (u) increases with the degree of ownership, especially when products are more independent (lower r). The reduced substitutability (lower r) at downstream level amplifies the effect of ownership on the input price because downstream firms, facing weaker competition, exert greater monopolistic power. This enables them to sell lower quantities at higher prices. Consequently, the reduced output by downstream firms leads to a lower input demand. In response to the lower input demand, the upstream supplier reacts, by increasing the input price (u).

Downstream operational profits $\left(v_i^{B}, v_j^{B}\right)$ decrease with higher ownership (α) due to the rising input costs, while upstream operational profits v_u^{B} increase with α as the in- put price rises. This opposing relationship makes it essential to analyze total downstream profits, which are the focus for downstream firms.[12]

Although operational profits decline with α, total downstream profits increase, because higher ownership leads to greater internalization of the upstream firm's decisions.[13] This internalization amplifies the weight of upstream profits in the total downstream profits.

Furthermore, the opposing profit effects perfectly offset each other, leaving industry profits (v_{IND}^{B}) constant regardless of the ownership level. Similarly, consumer surplus and social welfare remain unaffected by variations in α.

In PVBO, downstream firms have a perceived lower marginal cost because a portion of the input price paid to the upstream monopolist is effectively returned to them through their share of the upstream firm's profits. This dynamic reduces the effective input cost for downstream firms.

As a consequence, as ownership (α) increases, the elasticity of input demand with respect to the input price becomes less responsive, a trend observed in both Cournot and Bertrand competition. This rigidity in input demand reflects the reduced sensitivity of downstream firms to changes in input prices as they internalize part of the upstream profits.

[12] This follows from

$$\Pi_i^{B} = \Pi_j^{B} = \frac{1}{4}(1-g)\frac{-3\alpha+2\alpha g+1}{(g+1)(g-2)^2(-\alpha+\alpha g+1)} + \alpha\left(\frac{-1}{2(g+1)(-\alpha+\alpha g+1)(g-2)}\right)$$
$$= \frac{\left(1-g+\alpha+3\alpha g-2\alpha g^2\right)}{4(\alpha g-\alpha+1)(g-2)^2(g+1)}$$

[13] Indeed,

$$\frac{\partial \left(\Pi_i^{B}\right)}{\partial \alpha} = \frac{\partial \left(\Pi_j^{B}\right)}{\partial \alpha} = \frac{1}{4(\alpha g-\alpha+1)^2(2-g)} > 0$$

The relationship between ownership and demand elasticities are expressed as:

$$\frac{6\square o^{C\square}_{m}}{6\alpha} = -\frac{u}{(-u+u\alpha+1)^2} < 0$$

$$\frac{6\square o^{B\square}_{m}}{6\alpha} = -u\frac{1-r}{(u-u\alpha+u\alpha r-1)^2} < 0$$

Proposition 1 (PVBO): Under PVBO, input price (u) is higher under Gournot competition ($u^{C\square} > u^{B\square}$):

$$u^{C\square} - u^{B\square} = \frac{r\alpha}{2(\alpha r-\alpha+1)(1-\alpha)} > 0$$

This difference arises from differences in the elasticity of input demand, which is lower under Cournot compared to Bertrand competition ($\square o^{B\square}_m > \square o^{C\square}_m$) .

$$\square o^{C\square}_{m} - \square o^{B\square}_{m} = -\frac{u\alpha r}{(1-u+u\alpha)(1-u+u\alpha(1-r))} < 0$$

Input demand is more elastic under Bertrand competition ($\square o^{B\square}_m > \square o^{C\square}_m$) because down-stream firms compete on price, and even small increases in the input price (u) directly impact their ability to remain competitive.

In Bertrand competition, input demand is more elastic because downstream firms set prices closer to marginal cost, making the quantity demanded more sensitive to changes in u. As a result, an increase in the input price, u, directly raises output prices for consumers, leading to a more substantial reduction in the quantity demanded. This effect is particularly pronounced as the degree of product substitutability (r) increases, amplifying the sensitivity of demand to price changes.

In contrast, under Cournot competition, as the elasticity of demand ($\square o^{C\square}_m$) is lower, the downstream firms absorb a larger share of the cost increase, and when u increases the decrease on output is weaker. The rigidity of input demand under Cournot competition enables the upstream supplier to set a higher input price ($u^{C\square}$) without as large a loss in sales.

For all parameter values, the input demand is more rigid under quantity competition ($\square o^{B\square}_m > \square o^{C\square}_m$). As a consequence, the upstream is able to set a higher input price under quantity competition ($u^{C\square} > u^{B\square}$) .

Furthermore, as ownership (α) increases, the gap between input prices widens. The partial derivative of $(u^{C\square} - u^{B\square})$ with respect to α is positive:

$$\frac{\partial \left(u^{C\square} - u^{B\square}\right)}{\partial \alpha} = \frac{1}{2}\frac{(\alpha^2 r - \alpha^2 + 1)\, r}{(\alpha r - \alpha + 1)^2 (\alpha - 1)^2} > 0$$

This result indicates that the upstream supplier's ability to exploit the rigidity of input demand under Cournot competition becomes stronger as ownership increases, reinforcing the higher input price difference observed in this setting.

Proposition 2 (PVBO): Under PVBO the domnstream firms and the upstream firm might have higher profits mhen they compete on quantities or in prices (the result depends on the parameters: α and r). fhe total industry profits, consumer surplus, and social melfare, are higher mhen firms compete on prices.

In this setting, downstream firms face a complex set of incentives. While they gen- erally prefer lower input prices to boost operational profits, their ownership stake in the upstream firm makes higher input prices somewhat desirable as these prices contribute to the upstream's profits, of which they capture a portion. This internalization of upstream profits makes downstream firms more willing to tolerate higher input prices, especially when α is higher, offsetting some of their competitive instincts.

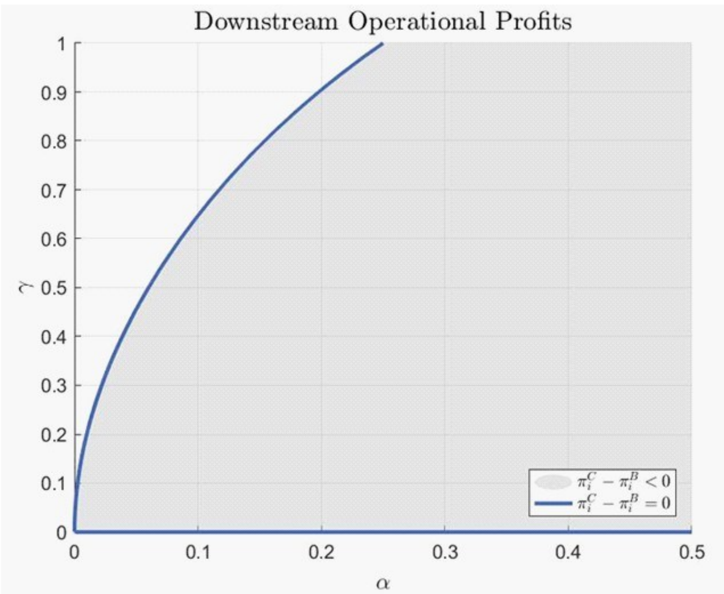

Figure 1: Downstream Operational Profits

Figure 1 illustrates the difference in downstream operational profits between Cournot and Bertrand ($v_i^C - v_i^B$). The solid blue line, represents all the points where the downstream profits are equal in the two types of competition ($v_i^C = v_i^B$)[14], while the grey area corresponds to all the cases where, given some values of α and r, price competition is always more profitable than quantity competition ($v_i^C - v_i^B < 0$).

Downstream profitability under Bertrand competition tends to be higher, when r is low (more independent products) and α high. The main driver behind the profit reversal outcome lies in the differences in input prices, where ($u^{C*} > u^{B*}$). If input prices were the same under both competition types, profit reversal would not occur.

When r is low, products are more independent, and the input price gap between Cournot and Bertrand ($u^{C*} - u^{B*}$) becomes smaller, though it remains positive. At the same time, the quantity produced under Cournot competition increases as r decreases. This rise in output intensifies the negative effect of the higher input price under Cournot competition, because the increased cost is being multiplied by a larger quantity. As a result, the traditional positive profit gap between Cournot and Bertrand ($v_i^C - v_i^B$) becomes smaller when r is low.

Additionally, a significant α (higher ownership) increases the input price gap ($u^{C*} - u^{B*}$), which directly affects the profitability differences between the downstream firms. All these effects contribute to making Bertrand more profitable for downstream firms in this setting ($v_i^C < v_i^B$).

[14] A particular case of this equality in profitability exists when g = 0, as visualized by the horizontal line on the x-axis. In this scenario, each downstream firm produces perfectly independent goods, corresponding to two monopolists operating in separate markets. Therefore, when g = 0, $\pi_i^C = \pi_i^B = \frac{1}{4}$, regardless of the value of □.

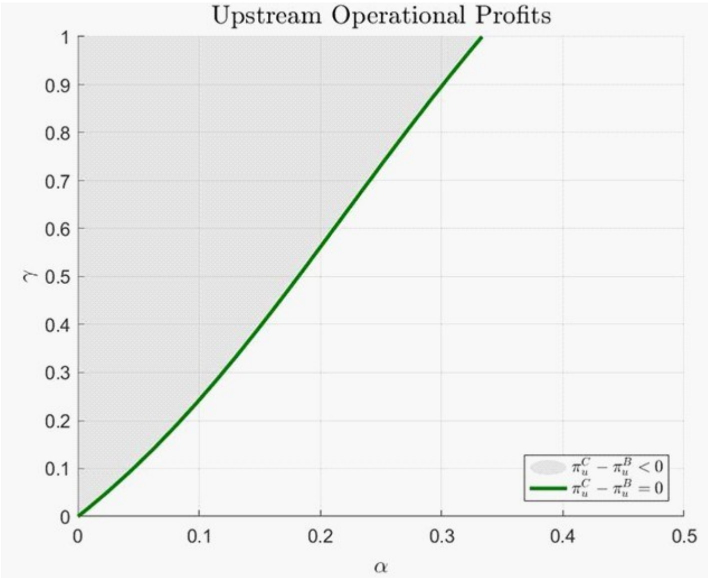

Figure 2: Upstream Operational Profits

Figure 2 represents the difference in upstream operational profits between Cournot and Bertrand ($v_u^C — v_u^B$). The green line corresponds to all the points where the upstream profits are equal in the two types of competition ($v_u^C = v_u^B$). The grey area here also represents all the cases where price competition is more profitable than quantity competition ($v_u^C — v_u^B < 0$).

For the upstream firm, profitability is generally higher under Bertrand when α is low (minimal ownership) and r is high (closer substitute products). Intuitively, the input price is lower under Bertrand competition ($u^{C\square} — u^{B\square} > 0$), and the input price difference decreases with lower α, eroding the upstream firm's advantage in Cournot competition as α diminishes.

When the ownership is smaller, downstream firms are more sensitive to upstream in- put price levels since they do not internalize the upstream profits as much. In Bertrand competition, the upstream firm sets a lower input price to support competitive pricing at the downstream level because that leads to an increase of downstream output, and consequentially, an increase of the upstream profits. When products are close substitutes, Bertrand competition intensifies, driving up demand at lower prices, which benefits the upstream firm through increased volume even at a lower input price. Under these conditions, low α and high r, the increase in quantity demanded under Bertrand outweighs the lower input price, leading to higher upstream profitability under Bertrand competi- tion. And, as a consequence, downstream firms become more competitive, which implies a higher output and, consequently a higher input production, benefitting the upstream supplier.

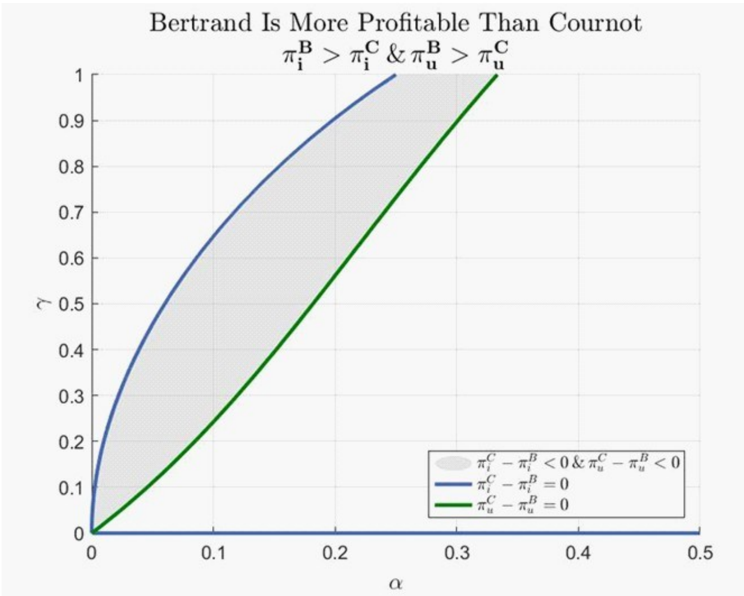

Figure 3: Operational Profit Reversal

Figure 3 represents the area of intersection where both downstream and upstream firms have higher profits under Bertrand competition. The grey area includes all the points where this is true, through the combination of ownership (α) and product differentiation (r). Although $\alpha \in [0; \frac{1}{2}]$, this result is only feasible when $\alpha \in [0; 0.325]$.

As expected, downstream decisions are based on total profits, which, under this own- ership structure, include a share of upstream profits. This alters downstream firms' incen- tives and introduces the possibility of profit reversals, making total downstream profits (N_i, N_j) essential for understanding their strategic choices.

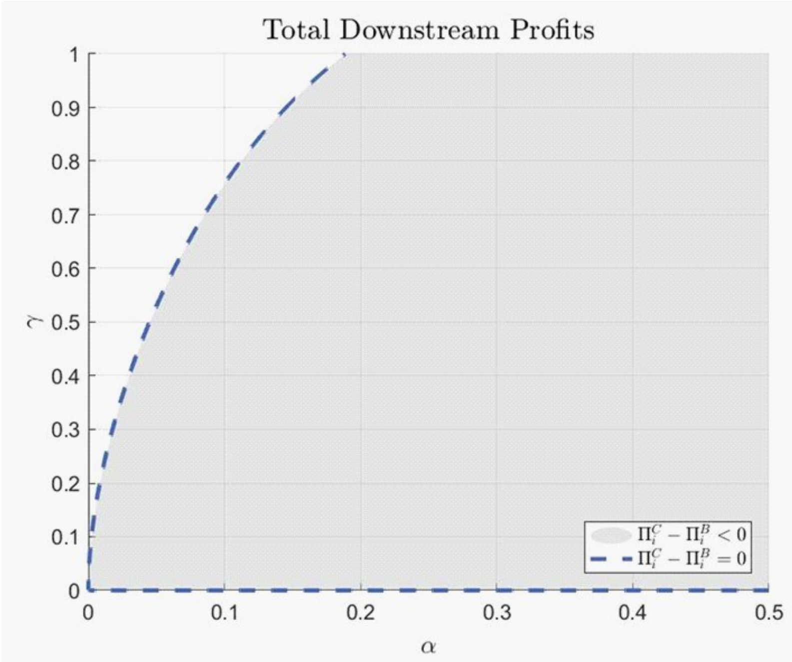

Figure 4: Total Downstream Profits

Figure 4 represents the difference on total downstream profits between Cournot and Bertrand $(\Pi_i^C - \Pi_i^B)$. The blue dashed line represents all points where Cournot and Bertrand competition yield equal profitability $(\Pi_i^C = \Pi_i^B)$, while the grey area highlights where, given some values of α and r, price competition is more profitable than quantity competition $(\Pi_i^C - \Pi_i^B < 0)$. Compared to the graph of operational downstream profits, this figure shows that the profit reversal occurs over a broader range of α and r values. Including the upstream share in total profits extends the reversal to lower α and higher r values than in the case of operational downstream profits alone, making Bertrand more advantageous in more scenarios.

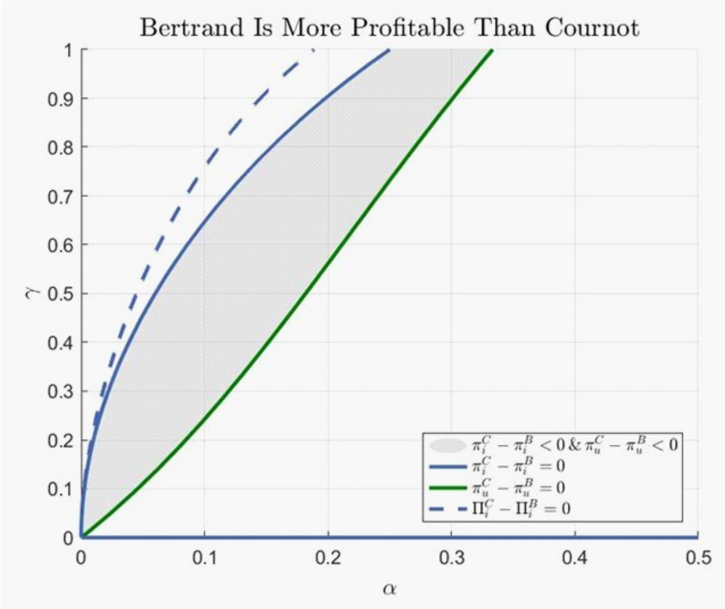

Figure 5: Total Profit Reversal

Figure 5 combines the insights from Figure 3 and Figure 4. It includes the opera- tional downstream and upstream profit curves from Figure 3 and incorporates the total downstream profit curve from Figure 4. The grey area, inherited from Figure 3, marks the region where Bertrand competition is more profitable than Cournot for both downstream and upstream operational profits.

However, Figure 5 reveals an additional and significant region: the area between the solid blue line and the dashed blue line. In this region, while operational downstream profits are higher under Cournot competition, total downstream profits are higher under Bertrand competition. Here, profit reversal arises because upstream profits are higher under Bertrand competition, and downstream firms internalize a portion of these up- stream profits through their ownership stake (α). As a result, total downstream profits favor Bertrand competition, even though Cournot competition retains the operational advantage.

This analysis extends the findings of Mukherjee et al (2024), who focus exclusively on operational profits. By introducing total downstream profits, it provides insight into the incentives of downstream firms, as they base their competitive decisions on total profits, rather than solely operational profits.

Consumer surplus is higher in Bertrand, because quantity is higher and prices are lower, which is aligned with the traditional cases (Singh & Vives 1984). Furthermore, Bertrand competition provides higher social welfare than Cournot competition because the consumer surplus is higher at Bertrand as well as the total industry profits.

$$q_i^{C} - q_j^{B} = \frac{r^2}{2(r+1)(r+2)(r-2)} < 0$$

$$P_i^{C} - P_j^{B} = \frac{r^2}{2(r+2)(2-r)} > 0$$

Greenlee and Raskovich (2006) provide results consistent with these findings in PVBO, demonstrating that in cases of symmetric ownership, upstream firms exploit the situation by raising input prices. This opportunistic behavior reduces competitive pressure among downstream firms, as they internalize a portion of the upstream profits. Consequently, the equilibrium input price rises with the level of aggregate ownership, benefitting the upstream supplier, at the expense of downstream firms' profitability.

The authors identify a property of strong invariance in symmetric backward ownership scenarios. They demonstrate that when downstream firms have symmetric stakes in an upstream supplier, equilibrium outcomes such as: equilibrium quantities and final prices, aggregate profits, consumer surplus, and social welfare are invariant, regardless of the aggregate ownership level (O) or the individual shares of ownership. This invariance arises because symmetric ownership profiles ensure that neither input costs nor strategic interactions among downstream firms are affected by variations in ownership.

Similarly, in this paper, this property persists within PVBO. Due to the symmetric ownership structure in our model, where ($\alpha_i = \alpha_j$), the individual share (α) and the aggregate ownership level (O) are equivalent, with $O = 2\alpha$. Consequently, the invariance observed in our results aligns directly with the findings of Greenlee and Raskovich (2006).

Mukherjee et al (2024) find that input prices tend to be lower under Bertrand com- petition, due to the higher input elasticity of the input demand in price competition. For final good producer, profits are higher under Bertrand competition when cross-ownership (α) is high and products are highly differentiated (r is low, meaning products are less substitutable). Conversely, profits for the upstream supplier are higher under Bertrand competition when cross-ownership is low (α is small) and product differentiation is low (r is high, meaning products are close substitutes). Moreover, their analysis reveals that industry profits, consumer surplus, and social welfare are consistently higher under Bertrand competition compared to Cournot competition.

Interestingly, the same results were observed in the case of PVBO alone, indicating that the inclusion of HCO and PVFO do not significantly affect the key outcomes of profit reversal, input pricing, profits, and welfare. Thus, the primary driver of the results in their paper is the scenario of PVBO.

By extending the analysis to total downstream profits (N_i, N_j), Bertrand competition outperforms Cournot over a wider range of of α and r values, as it includes cases where Bertrand yields higher total downstream profits, even though Cournot remains operationally more profitable for downstream firms.

5 Conclusion

This thesis examines three distinct symmetric ownership structures: Horizontal Cross-Ownership (HCO), Partial Vertical Forward Ownership (PVFO), and Partial Vertical Backward Ownership (PVBO), on market outcomes under Cournot and Bertrand competition. By focusing on the implications for firms' profitability, consumer surplus, and social welfare, this study provides valuable insights into how symmetric ownership struc- tures influence competition and the broader economy.

Social welfare maximization is a central goal of economic analysis, as it reflects the collective benefit to society. While Cournot competition is traditionally viewed as more profitable for firms due to less aggressive competition, Bertrand competition incentivizes more competitive outcomes. As a result, Bertrand competition often produces greater consumer surplus and higher social welfare. Under the ownership structures developed along this paper, Bertrand competition always delivers higher consumer surplus and social welfare, translating in more competitive outcomes, higher quantities and lower prices, compared with Cournot competition.

The PVBO case stands out as it identifies conditions under which Bertrand competi- tion also becomes more profitable for downstream firms. Specifically, when the ownership share (α) is high and the degree of product differentiation (r) is low, profit reversal occurs, challenging the conventional assumption that Cournot competition is inherently more advantageous for firms. This result is significant because it aligns the incentives of downstream firms and of the industry, with the interests of consumers.

This profit reversal is driven by the interaction between financial and operational incentives in the PVBO configuration. Downstream firms' partial ownership of the upstream supplier leads to internalization of upstream profits, altering their strategic responses to input prices. As a result, input prices are more rigid under Cournot competition, while Bertrand competition fosters higher output and lower input prices, ultimately benefiting downstream firms under certain conditions.

The findings of this thesis also align with Mukherjee et al (2024), who demonstrated the potential for Bertrand competition to deliver higher downstream and industry profits, consumer surplus, and welfare under cross-ownership. While their model relied on intri- cate ownership networks, this thesis shows that simplified configurations, such as PVBO alone, can replicate and clarify these effects. This underscores the relevance of ownership structure design in determining market outcomes.

Furthermore, this thesis explores the impact of ownership structures on total down- stream profits, the measure firms use to make decisions based on profitability, extending previous work. In the cases of HCO and PVFO, total downstream profits align closely with operational profits. However, under PVBO, this alignment no longer holds. Total downstream profits include a share of the upstream profits, altering the competitive dy- namics as downstream firms consider both their operational outcomes and the financial gains from upstream profitability. This redistribution dynamic can make Bertrand com- petition more profitable in total terms, even when Cournot competition remains superior in terms of operational profits.

Extending this analysis could provide even deeper insights. For instance, introducing a third stage in the model where firms endogenously choose their ownership levels, as explored in Yi Li (2024). Furthermore, allowing firms to select their strategic variables (prices or quantities) in an initial stage would offer a more realistic depiction of competitive dynamics, as these decisions fundamentally influence final outcomes. These extensions would enhance the applicability of the model to real-world scenarios, where firms decide their ownership and competitive strategies.

Finally, this research has practical implications for policy and regulation. By iden- tifying the conditions under which Bertrand competition harmonizes firm and consumer interests, it provides a foundation for antitrust authorities to assess ownership configu- rations and their potential impacts on market efficiency and equity. Policymakers could leverage these insights to monitor ownership shares and promote competitive conditions that maximize welfare, without stifling firm incentives.

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Appendix

Although total profits (N) are maximized, we focus on calculating the operational profits (v) for the downstream and the upstream firm, which will serve as the basis for our comparisons.

The operational profits are as follows:

Under Cournot:

$$v_i = (1 - q_i - rq_j - u)\, q_i \quad v_j = (1 - q_j - rq_i - u)\, q_j \quad v_u = u(q_i + q_j)$$

Under Bertrand:

$$v_i = (P_i - u)\frac{1}{r^2 - 1}(r + P_i - rP_j - 1)$$
$$v_j = (P_j - u)\frac{1}{r^2 - 1}(r + P_j - rP_i - 1)$$
$$v_u = u(\frac{1}{r^2 - 1}(r + P_i - rP_j - 1) + \frac{1}{r^2 - 1}(r + P_j - rP_i - 1))$$

Horizontal Cross-Ownership

Proof of Lemma 1 (HCO):

In this case, the objective functions maximized by the three firms are:

$$N_i = (1 - q_i - rq_j - u)\, q_i (1 - \alpha) + \alpha (1 - q_j - rq_i - u)\, q_j$$
$$N_j = (1 - q_i - rq_j - u)\, q_i \alpha + (1 - \alpha)(1 - q_j - rq_i - u)\, q_j$$
$$N_u = u(q_i + q_j)$$

At the second stage, when downstream firms choose output, the first-order conditions (FOC) are:

$$\frac{6\,((1 - q_i - rq_j - u)\, q_i (1 - \alpha) + \alpha (1 - q_j - rq_i - u)\, q_j)}{6q_i} = 0$$
$$\frac{6\,((1 - q_i - rq_j - u)\, q_i \alpha + (1 - \alpha)(1 - q_j - rq_i - u)\, q_j)}{6q_j} = 0$$

that result in:

$$q_i = (1-u)\frac{1-\alpha}{2-2\alpha+r}$$
$$q_j = (1-u)\frac{1-\alpha}{2-2\alpha+r}$$
$$Q = 2(1-u)\frac{1-\alpha}{2-2\alpha+r}$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{6\left(u2(1-u)\frac{1-\alpha}{2-2\alpha+r}\right)}{6u} = 0$$

From there the equilibrium input price:

$$u^{C} = \frac{1}{2}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$q_i^{C} = \frac{1}{2}\frac{1-\alpha}{2-2\alpha+r}$$
$$P_i^{C} = \frac{r-3\alpha+\alpha r+3}{2(r-2\alpha+2)}$$
$$v_i^{C} = \frac{1}{4}(1-\alpha)\frac{1-\alpha+\alpha r}{(2\alpha-r-2)^2}$$
$$v_j^{C} = \frac{1}{4}(1-\alpha)\frac{1-\alpha+\alpha r}{(2\alpha-r-2)^2}$$
$$v_u^{C} = \frac{1-\alpha}{2(r-2\alpha+2)}$$
$$v_{IND}^{C} = \frac{(r-3\alpha+\alpha r+3)(1-\alpha)}{2(r-2\alpha+2)^2}$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2} = 2q - q^2(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$
\begin{aligned}
CS^{C*} &= 2\left(\frac{1}{2}\frac{1-\alpha}{2-2\alpha+r}\right) - \left(\frac{1}{2}\frac{1-\alpha}{2-2\alpha+r}\right)^2(1+r) \\
&\quad -2\left(\frac{1}{2}\frac{1-\alpha}{2-2\alpha+r}\right)\left(\frac{r-3\alpha+\alpha r+3}{2(r-2\alpha+2)}\right) \\
&= \frac{(r+1)(\alpha-1)^2}{4(r-2\alpha+2)^2} \\
SW^{C*} &= CS^{C*} + v^{C*}_{IND} = \frac{(r+1)(\alpha-1)^2}{4(r-2\alpha+2)^2} + \frac{(r-3\alpha+\alpha r+3)(1-\alpha)}{2(r-2\alpha+2)^2} \\
&= \frac{(3r-7\alpha+\alpha r+7)(1-\alpha)}{4(r-2\alpha+2)^2}
\end{aligned}
$$

Auxiliary derivatives:

$$
\begin{aligned}
\frac{\partial q_i^{C*}}{\partial \alpha} &= -\frac{1}{2}\frac{r}{(2\alpha-r-2)^2} < 0 \\
\frac{\partial\left(-\frac{1}{2}\frac{r}{(2\alpha-r-2)^2}\right)}{\partial r} &= \frac{1}{2}\frac{2\alpha+r-2}{(r-2\alpha+2)^3} < 0 \\
\frac{\partial v_i^{C*}}{\partial \alpha} &= \frac{1}{4}r^2\frac{1-2\alpha}{(2-2\alpha+r)^3} > 0 \\
\frac{\partial v_u^{C*}}{\partial \alpha} &= -\frac{1}{2}\frac{r}{(2\alpha-r-2)^2} < 0 \\
\frac{\partial v_{IND}^{C*}}{\partial \alpha} &= r\frac{-\alpha+\alpha r+1}{(2\alpha-r-2)^3} < 0 \\
\frac{\partial CS^{C*}}{\partial \alpha} &= \frac{1}{2}r(1-\alpha)\frac{r+1}{(2\alpha-r-2)^3} < 0 \\
\frac{\partial SW^{C*}}{\partial \alpha} &= \frac{1}{2}\frac{(r-3\alpha+\alpha r+3)r}{(2\alpha-r-2)^3} < 0
\end{aligned}
$$

Proof of Lemma 2 (HCO):

In this case, the objective functions maximized by the three firms are:

$$
\begin{aligned}
\Pi_i &= (P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)(1-\alpha) + \alpha(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1) \\
\Pi_j &= (P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)\alpha + (1-\alpha)(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1) \\
\Pi_u &= u\left(\frac{1}{r^2-1}(r+P_i-rP_j-1) + \frac{1}{r^2-1}(r+P_j-rP_i-1)\right)
\end{aligned}
$$

At the second stage, when downstream choose output, the first-order conditions (FOC) are:

$$\frac{\partial\left[(P_i - u)\frac{1}{r^2-1}(r + P_i - rP_j - 1)(1-\alpha) + \alpha(P_j - u)\frac{1}{r^2-1}(r + P_j - rP_i - 1)\right]}{\partial P_i} = 0$$

$$\frac{\partial\left[(P_i - u)\frac{1}{r^2-1}(r + P_i - rP_j - 1)\alpha + (1-\alpha)(P_j - u)\frac{1}{r^2-1}(r + P_j - rP_i - 1)\right]}{\partial P_j} = 0$$

that result in:

$$P_i = \frac{-u + \alpha + r + u\alpha - \alpha r + u\alpha r - 1}{2\alpha + r - 2}$$

$$P_j = \frac{-u + \alpha + r + u\alpha - \alpha r + u\alpha r - 1}{2\alpha + r - 2}$$

$$Q = \left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right) = 2(1-u)\frac{1-\alpha-\alpha r}{(r+1)(2-2\alpha-r)}$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{\partial\left[u\left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right)\right]}{\partial u} = 0$$

From there the equilibrium input price:

$$u^{B*} = \frac{1}{2}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$q_i^{B*} = \frac{1}{2}\frac{1-\alpha-\alpha r}{(r+1)(2-2\alpha-r)}$$

$$P_i^{B*} = \frac{\alpha r - 2r - 3\alpha + 3}{2(2-2\alpha-r)}$$

$$v_i^{B\square} = \frac{(1-\alpha)(r-1)(\alpha+\alpha r-1)\,4}{(2\alpha+r-2)^2\,(r+1)}$$

$$v_j^{B\square} = \frac{(1-\alpha)(r-1)(\alpha+\alpha r-1)\,4}{(2\alpha+r-2)^2\,(r+1)}$$

$$v_u^{B\square} = \frac{\alpha+\alpha r-1}{2\,(2\alpha+r-2)\,(r+1)}$$

$$v_{IND}^{B\square} = \frac{(1-\alpha-\alpha r)\,(\alpha r-2r-3\alpha+3)\,2}{(2\alpha+r-2)^2\,(r+1)}$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2} = 2q - q^2\,(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2\,(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$CS^{B\square} = 2\left(\frac{1}{2\,(r+1)}\,\frac{1-\alpha-\alpha r}{(2-2\alpha-r)}\right) - \left(\frac{1}{2\,(r+1)}\,\frac{1-\alpha-\alpha r}{(2-2\alpha-r)}\right)^2 (1+r)$$
$$-2\left(\frac{1}{2\,(r+1)}\,\frac{1-\alpha-\alpha r}{(2-2\alpha-r)}\right)\left(\frac{\alpha r-2r-3\alpha+3}{2\,(2-2\alpha-r)}\right)$$
$$= \frac{(\alpha+\alpha r-1)^2}{4\,(2\alpha+r-2)^2\,(r+1)}$$

$$SW^{B\square} = CS^{B\square} + v_{IND}^{B\square} = \frac{(\alpha+\alpha r-1)^2}{4\,(2\alpha+r-2)^2\,(r+1)} + \frac{(1-\alpha-\alpha r)\,(\alpha r-2r-3\alpha+3)}{2\,(2\alpha+r-2)^2\,(r+1)}$$
$$= \frac{(1-\alpha-\alpha r)\,(\alpha r-4r-7\alpha+7)}{4\,(2\alpha+r-2)^2\,(r+1)}$$

Auxiliary derivatives:

$$
\begin{aligned}
\frac{\partial P_i^{B*}}{\partial \alpha} &= \frac{1}{2}\frac{(1-r)\,r}{(2\alpha+r-2)^2} > 0 \\
\frac{\partial\left(\frac{1}{2}\frac{(1-r)r}{(2\alpha+r-2)^2}\right)}{\partial r} &= -\frac{1}{2}\frac{4\alpha r-3r-2\alpha+2}{(2\alpha+r-2)^3} > 0 \\
\frac{\partial v_i^{B*}}{\partial \alpha} &= -\frac{1}{4}\frac{(1-r)(1-2\alpha)\,r^2}{(2\alpha+r-2)^3\,(r+1)} > 0 \\
\frac{\partial v_u^{B*}}{\partial \alpha} &= -\frac{1}{2}\frac{(1-r)\,r}{(2\alpha+r-2)^2\,(r+1)} < 0 \\
\frac{\partial v_{IND}^{B*}}{\partial \alpha} &= -\frac{(1-\alpha)(r-1)^2\,r}{(2-r-2\alpha)^3\,(r+1)} < 0 \\
\frac{\partial CS^{B*}}{\partial \alpha} &= \frac{1}{2}\frac{(\alpha+\alpha r-1)(r-1)\,r}{(2\alpha+r-2)^3\,(r+1)^1} < 0 \\
\frac{\partial SW^{B*}}{\partial \alpha} &= \frac{1}{2}\frac{(\alpha r-2r-3\alpha+3)(1-r)\,r}{(2\alpha+r-2)^3\,(r+1)} < 0
\end{aligned}
$$

Proof of Proposition 1 (HCO):

$$u^{C*} - u^{B*} = \frac{1}{2} - \frac{1}{2} = 0$$

The wage is the same in Cournot and Bertrand because ownership happens at the downstream level, and the type of competition does not change the incentives of the upstream.

Input demands:

$$
\begin{aligned}
Q^C(u) &= 2\,(1-u)\,\frac{1-\alpha}{2-2\alpha+r} \\
Q^B(u) &= -2\,\frac{(u-1)(\alpha+\alpha r-1)}{(2\alpha+r-2)(r+1)}
\end{aligned}
$$

Ownership has a negative impact on input demands:

$$
\begin{aligned}
\frac{\partial Q^C(u)}{\partial \alpha} &= \frac{2\,(u-1)\,r}{(2\alpha-r-2)^2} < 0 \\
\frac{\partial Q^B(u)}{\partial \alpha} &= -2\,\frac{(1-r)(1-u)\,r}{(2\alpha+r-2)^2\,(r+1)} < 0
\end{aligned}
$$

Elasticities:

$$o^C = \frac{6 \; 2(1-u)\; \frac{1-\square}{2-2\square+r}}{6u} \; \frac{u}{2(1-u)\;\frac{1-\square}{2-2\square+r}} = \frac{u}{u-1}$$

$$o^B = \frac{6 \; -2\;\frac{(m-1)(\square+\square r-1)}{(2\square+r-2)(r+1)}}{6u} \; \frac{u}{-2\;\frac{(m-1)(\square+\square r-1)}{(2\square+r-2)(r+1)}} = \frac{u}{u-1}$$

So, when $u^{C\square} = u^{B\square} = \frac{1}{2}$, $\square o^C\square = \square o^B\square = 1$.

Proof of Proposition 2 (HCO):

$$v_i^C - v_i^B = \frac{1}{4}(1-\alpha)\;\frac{1-\alpha+\alpha r}{(2\alpha-r-2)^2} - \frac{(1-\alpha)(r-1)(\alpha+\alpha r-1)}{4(2\alpha+r-2)^2(r+1)}$$

$$= \frac{r^3(1-\alpha)(2\alpha-1)^2}{2(r+1)(2\alpha-r-2)^2(2\alpha+r-2)^2} > 0$$

$$v_u^C - v_u^B = \frac{1-\alpha}{2(r-2\alpha+2)} - \frac{\alpha+\alpha r-1}{2(2\alpha+r-2)(r+1)}$$

$$= -\frac{r^2(1-2\alpha)}{2(r+1)(1-2\alpha+1-r)(2-2\alpha+r)} < 0$$

$$v_{IND}^C - v_{IND}^B = \frac{1}{2}r^2(1-2\alpha)\;\frac{-4\alpha^2(1-r)+2\alpha(4-3r)+2r+r^2-4}{(r+1)(2\alpha-r-2)^2(2\alpha+r-2)^2} < 0$$

$$CS^C - CS^B = \frac{(r+1)(\alpha-1)^2\,4}{(r-2\alpha+2)^2} - \frac{(\alpha+\alpha r-1)^2}{4(2\alpha+r-2)^2(r+1)}$$

$$= \frac{(2r-8\alpha-6\alpha r+4\alpha^2-r^2+4\alpha^2 r+4)(2\alpha-1)r^2}{4(2\alpha+r-2)^2(r-2\alpha+2)^2(r+1)} < 0$$

$$SW^C - SW^B = \square CS^C - CS^B\square + \square v_{IND}^C - v_{IND}^B\square$$

$$= \frac{(24\alpha+2r-6\alpha r-12\alpha^2+3r^2+4\alpha^2 r-12)(1-2\alpha)r^2}{4(2\alpha+r-2)^2(r-2\alpha+2)^2(r+1)} < 0$$

To explain welfare measures, let us see the difference in quantities and prices, between Cournot and Bertrand:

$$q_i^{C\square} - q_j^{B\square} = \frac{1}{2}\frac{1-\alpha}{2-2\alpha+r} - \frac{1}{2}\frac{1-\alpha-\alpha r}{(r+1)(2-2\alpha-r)}$$

$$= \frac{(1-2\alpha)r^2}{2(2\alpha-r-2)(2-r-2\alpha)(r+1)} < 0$$

$$P^{C}_{i} - P^{B}_{j} = \frac{r(\alpha+1)+3(1-\alpha)}{2(r+2(1-\alpha))} - \frac{\alpha r - 2r - 3\alpha + 3}{2(2-2\alpha-r)}$$
$$= \frac{(1-2\alpha)r^2}{2(2\alpha+r-2)(2\alpha-r-2)} > 0$$

Partial Vertical Forward Ownership

Proof of Lemma 1 (PVFO):

In this case, the objective functions maximized by the three firms are:

$$N_i = (1 - q_i - rq_j - u)\, q_i (1-\alpha)$$
$$N_j = (1-\alpha)(1 - q_j - rq_i - u)\, q_j$$
$$N_u = u(q_i + q_j) + \alpha(1 - q_i - rq_j - u)\, q_i + \alpha(1 - q_j - rq_i - u)\, q_j$$

At the second stage, when downstream choose output, the first-order conditions (FOC) are:

$$\frac{6((1 - q_i - rq_j - u)\, q_i (1-\alpha))}{6q_i} = 0$$
$$\frac{6((1-\alpha)(1 - q_j - rq_i - u)\, q_j)}{6q_j} = 0$$

that result in:

$$q_i = \frac{1}{r+2}(1-u)$$
$$q_j = \frac{1}{r+2}(1-u)$$
$$Q = \frac{2}{r+2}(1-u)$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{6\,(u(q_i + q_j) + \alpha(1 - q_i - rq_j - u)\, q_i + \alpha(1 - q_j - rq_i - u)\, q_j)}{6u} = 0$$

From there the equilibrium input price:

$$u^{C} = \frac{2 - 2\alpha + r}{4 - 2\alpha + 2r}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$
\begin{aligned}
q_i^{C\square} &= \frac{1}{2(2-\alpha+r)} \\
P_i^{C\square} &= \frac{r-2\alpha+3}{2(r-\alpha+2)} \\
v_i^{C\square} &= \frac{1}{4(\alpha-r-2)^2} = \frac{1}{4(r-\alpha+2)^2} \\
v_j^{C\square} &= \frac{1}{4(\alpha-r-2)^2} \\
v_u^{C\square} &= \frac{1}{2}\frac{2-2\alpha+r}{(\alpha-r-2)^2} = \frac{(r-2\alpha+2)}{2(r-\alpha+2)^2} \\
v_{IND}^{C\square} &= \frac{r-2\alpha+3\ 2}{(r-\alpha+2)^2}
\end{aligned}
$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2} = 2q - q^2(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$
\begin{aligned}
CS^{C\square} &= 2\left(\frac{1}{2(2-\alpha+r)}\right) - \left(\frac{1}{2(2-\alpha+r)}\right)^2(1+r) \\
&\quad -2\left(\frac{1}{2(2-\alpha+r)}\right)\left(\frac{r-2\alpha+3}{2(r-\alpha+2)}\right) \\
&= \frac{r+1}{4(\alpha-r-2)^2} \\
SW^{C\square} &= CS^{C\square} + v_{IND}^{C\square} = \frac{r+1}{4(\alpha-r-2)^2} + \frac{r-2\alpha+3}{2(r-\alpha+2)^2} = \frac{3r-4\alpha+7}{4(r-\alpha+2)^2}
\end{aligned}
$$

Auxiliary derivatives:

$$\frac{\partial\left(u^{C}\right)}{\partial\alpha} = -\frac{r+2}{2(\alpha-r-2)^2} < 0$$

$$\frac{\partial\left(-\frac{r+2}{2(\alpha-r-2)^2}\right)}{\partial r} = \frac{1}{2}\frac{\alpha+r+2}{(r-\alpha+2)^3} > 0$$

$$\frac{\partial\left(q_i^{C}\right)}{\partial\alpha} = \frac{1}{2(r-\alpha+2)^2} > 0$$

$$\frac{\partial\left(\frac{1}{2(r-\alpha+2)^2}\right)}{\partial r} = -\frac{1}{(r-\alpha+2)^3} < 0$$

$$\frac{\partial\left(v_i^{C}\right)}{\partial\alpha} = \frac{1}{2(r-\alpha+2)^3} > 0$$

$$\frac{\partial\left(v_u^{C}\right)}{\partial\alpha} = -\frac{\alpha}{(r-\alpha+2)^3} < 0$$

$$\frac{\partial\left(v_{IND}^{C}\right)}{\partial\alpha} = \frac{1-\alpha}{(r-\alpha+2)^3} > 0$$

$$\frac{\partial\left(CS^{C}\right)}{\partial\alpha} = \frac{r+1}{2(2-\alpha+r)^3} > 0$$

$$\frac{\partial\left(SW^{C}\right)}{\partial\alpha} = \frac{r-2\alpha+3}{2(r-\alpha+2)^3} > 0$$

Proof of Lemma 2 (PVFO):

In this case, the objective functions maximized by the three firms are:

$$\Pi_i = (P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)(1-\alpha)$$

$$\Pi_j = (1-\alpha)(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1)$$

$$\Pi_u = u\left(\frac{1}{r^2-1}(r+P_i-rP_j-1)+\frac{1}{r^2-1}(r+P_j-rP_i-1)\right)+ \alpha(P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)+\alpha(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1)$$

At the second stage, when downstream choose output, the first-order conditions (FOC) are:

$$\frac{\partial\left((P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)(1-\alpha)\right)}{\partial P_i} = 0$$

$$\frac{\partial\left((1-\alpha)(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1)\right)}{\partial P_j} = 0$$

that result in:

$$P_i = \frac{1}{2-r}(u-r+1)$$
$$P_j = \frac{1}{2-r}(u-r+1)$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{6\left[\begin{array}{c} u\left(\frac{1}{r^2-1}(r+P_i-rP_j-1)+\frac{1}{r^2-1}(r+P_j-rP_i-1)\right) \\ +\alpha(P_i-u)\frac{1}{r^2-1}(r+P_i-rP_j-1)+\alpha(P_j-u)\frac{1}{r^2-1}(r+P_j-rP_i-1)\end{array}\right]}{6u}=0$$

From there the equilibrium input price:

$$u^{B*} = \frac{1}{2}\frac{-2\alpha-r+2\alpha r+2}{-\alpha-r+\alpha r+2}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$q_i^{B*} = \frac{1}{2(r+1)(2-\alpha-r+\alpha r)}$$
$$P_i^{B*} = \frac{2\alpha r-2r-2\alpha+3}{2(\alpha r-r-\alpha+2)}$$
$$V_i^{B*} = \frac{1-r}{4(\alpha r-r-\alpha+2)^2(r+1)}$$
$$V_j^{B*} = \frac{1-r}{4(\alpha r-r-\alpha+2)^2(r+1)}$$
$$V_u^{B*} = \frac{2\alpha r-r-2\alpha+2}{2(\alpha r-r-\alpha+2)^2(r+1)}$$
$$V_{IND}^{B*} = \frac{2\alpha r-2r-2\alpha+3}{2(\alpha r-r-\alpha+2)^2(r+1)}$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2+q_j^2+2rq_iq_j}{2} = 2q - q^2(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$CS^{B*} = 2\left[\frac{1}{2(r+1)(2-\alpha-r+\alpha r)} - \frac{1}{2(r+1)(2-\alpha-r+\alpha r)}\right]^2 (1+r)$$

$$-2\left[\frac{1}{2(r+1)(2-\alpha-r+\alpha r)}\right]\left[\frac{2\alpha r-2r-2\alpha+3}{2(\alpha r-r-\alpha+2)}\right]$$

$$= \frac{1}{(r+1)(\alpha r-r-\alpha+2)^2}$$

$$SW^{B*} = CS^{B*} + v^{B*}_{IND} = \frac{1}{(r+1)(\alpha r-r-\alpha+2)^2} + \frac{2\alpha r-2r-2\alpha+3}{2(\alpha r-r-\alpha+2)^2(r+1)}$$

$$= \frac{2\alpha r-2r-2\alpha+5}{2(\alpha r-r-\alpha+2)^2(r+1)}$$

Auxiliary derivatives:

$$\frac{\partial u^{B*}}{\partial \alpha} = -\frac{1}{2}\frac{(2-r)(1-r)}{(\alpha r-r-\alpha+2)^2} < 0$$

$$\frac{\partial\left(-\frac{1}{2}\frac{(2-r)(1-r)}{(\alpha r-r-\alpha+2)^2}\right)}{\partial r} = -\frac{1}{2}\frac{r-\alpha+\alpha r-2}{(\alpha r-r-\alpha+2)^3} > 0$$

$$\frac{\partial P_i^{B*}}{\partial \alpha} = -\frac{1}{2}\frac{(1-r)}{(\alpha r-r-\alpha+2)^2} < 0$$

$$\frac{\partial\left(-\frac{1}{2}\frac{(1-r)}{(\alpha r-r-\alpha+2)^2}\right)}{\partial r} = -\frac{1}{2}\frac{(\alpha r-r-\alpha)}{(\alpha r-r-\alpha+2)^3} > 0$$

$$\frac{\partial v_i^{B*}}{\partial \alpha} = \frac{1}{2}\frac{(r-1)^2}{(\alpha r-r-\alpha+2)^3(r+1)} > 0$$

$$\frac{\partial v_U^{B*}}{\partial \alpha} = -\frac{(r-1)^2\alpha}{(\alpha r-r-\alpha+2)^3(r+1)} < 0$$

$$\frac{\partial v_{IND}^{B*}}{\partial \alpha} = \frac{(1-\alpha)(r-1)^2}{(\alpha r-r-\alpha+2)^3(r+1)} > 0$$

$$\frac{\partial CS^{B*}}{\partial \alpha} = -2\frac{(r-1)}{(\alpha r-r-\alpha+2)^3(r+1)} > 0$$

$$\frac{\partial SW^{B*}}{\partial \alpha} = \frac{(\alpha r-r-\alpha+3)(1-r)}{(\alpha r-r-\alpha+2)^3(r+1)} > 0$$

Proof of Proposition 1 (PVFO):

$$u^{C*} - u^{B*} = \frac{2-2\alpha+r}{4-2\alpha+2r} - \frac{1}{2}\frac{-2\alpha-r+2\alpha r+2}{-\alpha-r+\alpha r+2} = -\frac{r^2\alpha}{2(\alpha r-r-\alpha+2)(r-\alpha+2)} < 0$$

$$\frac{\partial\left(u^{C}-u^{B}\right)}{\partial\alpha}=\frac{\partial\left(-\frac{r^{2}}{2(\alpha r-r-\alpha+2)(r-\alpha+2)}\right)}{\partial\alpha}=-\frac{1}{2}\frac{(\alpha^{2}r-r^{2}-\alpha^{2}+4)\,r^{2}}{(\alpha r-r-\alpha+2)^{2}(\alpha-r-2)^{2}}<0$$

Auxiliary calculations to explain the input price difference:

$$N_u = uQ(u)+\alpha 2v_d(u)$$

$$\frac{\partial N_u}{\partial u} = Q(u)+u\frac{\partial Q}{\partial u}+2\alpha\frac{\partial v_d}{\partial u}$$

$$\frac{\partial v_i^C}{\partial u}-\frac{\partial v_i^B}{\partial u}=\frac{\partial\left(\frac{(1-m)^2}{(r+2)^2}\right)}{\partial u}-\frac{\partial\left((1-u)^2\frac{1-r}{(r+1)(r-2)^2}\right)}{\partial u}=-\frac{4r^3(1-u)}{(r+1)(2-r)^2(r+2)^2}<0$$

$$v_i^C = \left(1-\frac{1}{r+2}(1-u)-r\frac{1}{r+2}(1-u)-u\right)\frac{1}{r+2}(1-u)=\frac{(u-1)^2}{(r+2)^2}$$

$$v_i^B = \left(\frac{1}{2-r}(u-r+1)-u\right)\frac{1-u}{(r+1)(2-r)}=\frac{(1-r)(u-1)^2}{(r-2)^2(r+1)}$$

$$u^B > u^C$$

Proof of Proposition 2 (PVFO):

$$v_i^C-v_i^B = \frac{1}{4(\alpha-r-2)^2}-\frac{1}{4}\frac{1-r}{(-\alpha-r+\alpha r+2)^2(r+1)}$$

$$= \frac{1}{4}\frac{r^2(2\alpha(1-r)+2r+\alpha^2 r-\alpha^2)}{(r+1)(\alpha-r-2)^2(-\alpha-r+\alpha r+2)^2}>0$$

$$v_u^C-v_u^B = \frac{(r-2\alpha+2)}{2(r-\alpha+2)^2}-\frac{2\alpha r-r-2\alpha+2}{2(\alpha r-r-\alpha+2)^2(r+1)}$$

$$= -\frac{1}{2}r^2\frac{(7\alpha^2-2\alpha^3-r^2-\alpha^2r^2+4\alpha r+2\alpha r^2-5\alpha^2 r+2\alpha^3 r+4-8\alpha)}{(r+1)(\alpha-r-2)^2(-\alpha-r+\alpha r+2)^2}<0$$

$$v_{IND}^C-v_{IND}^B = \frac{1}{2}r^2(\alpha-1)^2\frac{2\alpha+2r-2\alpha r+r^2-4}{(r+1)(\alpha-r-2)^2(-\alpha-r+\alpha r+2)^2}<0$$

$$CS^C - CS^B = \frac{r+1}{4(\alpha - r - 2)^2} - \frac{1}{(r+1)(\alpha r - r - \alpha + 2)^2}$$

$$= \frac{(3r - 3\alpha - r^2 + \alpha r^2 + 6)(\alpha - r - r^2 + \alpha r^2 - 2)}{4(\alpha r - r - \alpha + 2)^2 (r - \alpha + 2)^2 (r+1)} < 0$$

$$SW^C - SW^B = \frac{3r - 4\alpha + 7}{4(r - \alpha + 2)^2} - \frac{2\alpha r - 2r - 2\alpha + 5}{2(\alpha r - r - \alpha + 2)^2 (r+1)}$$

$$= \frac{\alpha(12 - 3\alpha) + r(2r^2 + 3r^3 - 12 - 15r) - 12}{4(\alpha r - r - \alpha + 2)^2 (r - \alpha + 2)^2 (r+1)}$$

$$+ \frac{\alpha r(6 + 26r - 10r^2 - 6r^3 - 18\alpha r + 12\alpha r^2 + 4\alpha^2 r + 3\alpha r^3 - 4\alpha^2 r^2)}{4(\alpha r - r - \alpha + 2)^2 (r - \alpha + 2)^2 (r+1)} < 0$$

To explain welfare measures, let us see the difference in quantities and prices, between Cournot and Bertrand:

$$q_i^{C*} - q_j^{B*} = \frac{1}{2(2 - \alpha + r)} - \frac{1}{2(r+1)(2 - \alpha - r + \alpha r)}$$

$$= \frac{(\alpha - 1) r^2}{2(\alpha - r - 2)(\alpha + r - \alpha r - 2)(r+1)} < 0$$

$$P_i^{C*} - P_j^{B*} = \frac{r - 2\alpha + 3}{2(r - \alpha + 2)} - \frac{2\alpha r - 2r - 2\alpha + 3}{2(\alpha r - r - \alpha + 2)} = \frac{(\alpha - 1) r^2}{2(\alpha r - r - \alpha + 2)(\alpha - r - 2)} > 0$$

Partial Vertical Backwards Ownership

Proof of Lemma 1 (PVBO):

In this case, the objective functions maximized by the three firms are:

$$N_i = (1 - q_i - rq_j - u)\, q_i + \alpha u(q_i + q_j)$$
$$N_j = (1 - q_j - rq_i - u)\, q_j + \alpha u(q_i + q_j)$$
$$N_u = u(q_i + q_j)(1 - 2\alpha)$$

At the second stage, when downstream choose output, the first-order conditions (FOC) are:

$$\frac{6((1 - q_i - rq_j - u)\, q_i + \alpha u(q_i + q_j))}{6q_i} = 0$$
$$\frac{6((1 - q_j - rq_i - u)\, q_j + \alpha u(q_i + q_j))}{6q_j} = 0$$

that result in:

$$q_i = \frac{-u + u\alpha + 1}{r + 2}$$

$$q_j = \frac{-u + u\alpha + 1}{r + 2}$$

$$Q = 2\frac{-u + u\alpha + 1}{r + 2}$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{6\left(u2\frac{-m+m\alpha+1}{r+2} \right)}{6u} = 0$$

From there the equilibrium input price:

$$u^{C} = \frac{1}{2(1-\alpha)}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$q_i^{C} = \frac{1}{2(r+2)}$$

$$P_i^{C} = 1-(1+r)\left(\frac{1}{2(r+2)}\right) = \frac{r+3}{2(r+2)}$$

$$v_i^{C} = \frac{1}{4}\frac{3\alpha+\alpha r-1}{(r+2)^2(\alpha-1)}$$

$$v_j^{C} = \frac{1}{4}\frac{3\alpha+\alpha r-1}{(r+2)^2(\alpha-1)} = \frac{1-3\alpha-\alpha r}{4(1-\alpha)(r+2)^2}$$

$$v_u^{C} = -\frac{1}{2(r+2)(\alpha-1)} = \frac{1}{2(1-\alpha)(r+2)}$$

$$v_{IND}^{C} = \frac{1}{2}\frac{r+3}{(r+2)^2}$$

$$N_i^{C} = \frac{1}{4}\frac{3\alpha+\alpha r-1}{(r+2)^2(\alpha-1)} + \frac{\alpha}{2(1-\alpha)(r+2)} = \frac{(\alpha+\alpha r+1)}{4(1-\alpha)(r+2)^2}$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2} = 2q - q^2(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$
\begin{aligned}
CS^{C*} &= 2\left(\frac{1}{2(r+2)}\right) - \left(\frac{1}{2(r+2)}\right)^2(1+r) - 2\left(\frac{1}{2(r+2)}\right)\left(\frac{r+3}{2(r+2)}\right) \\
&= \frac{(r+1)}{4(r+2)^2} \\
SW^{C*} &= CS^{C*} + v^{C*}_{IND} = \frac{(r+1)}{4(r+2)^2} + \frac{1}{2}\frac{r+3}{(r+2)^2} = \frac{3r+7}{4(r+2)^2}
\end{aligned}
$$

Auxiliary derivatives:

$$
\begin{aligned}
\frac{\partial\left(u^{C*}\right)}{\partial\alpha} &= \frac{1}{2(\alpha-1)^2} > 0 \\
\frac{\partial\left(q_i^{C*}\right)}{\partial\alpha} &= 0 \\
\frac{\partial\left(q_i^{C*}\right)}{\partial r} &= -\frac{1}{2(r+2)^2} < 0 \\
\frac{\partial\left(v_i^{C*}\right)}{\partial\alpha} &= -\frac{1}{4(r+2)(\alpha-1)^2} < 0 \\
\frac{\partial\left(v_u^{C*}\right)}{\partial\alpha} &= \frac{1}{2(\alpha-1)^2(r+2)} > 0 \\
\frac{\partial\left(v_{IND}^{C*}\right)}{\partial\alpha} &= 0 \\
\frac{\partial\left(N_i^{C*}\right)}{\partial\alpha} &= \frac{1}{4(\alpha-1)^2(r+2)} > 0 \\
\frac{\partial CS^{C*}}{\partial\alpha} &= 0 \\
\frac{\partial SW^{C*}}{\partial\alpha} &= 0
\end{aligned}
$$

Proof of Lemma 2 (PVBO):

In this case, the objective functions maximized by the three firms are:

$$
\begin{aligned}
N_i &= (P_i - u)\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \\
&\quad + \alpha u\left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right) \\
N_j &= (P_j - u)\frac{1}{r^2-1}(r + P_j - rP_i - 1) + \\
&\quad + \alpha u\left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right)
\end{aligned}
$$

$$N_u = u\left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right)(1-2\alpha)$$

At the second stage, when downstream choose output, the first-order conditions (FOC) are:

$$\frac{\partial N_i}{\partial P_i} = 0$$

$$\frac{\partial N_j}{\partial P_j} = 0$$

that result in:

$$P_i = -\frac{1}{r-2}(u - r - u\alpha + u\alpha r + 1)$$

$$P_j = -\frac{1}{r-2}(u - r - u\alpha + u\alpha r + 1)$$

$$Q = \left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right) = 2\frac{1 - u(1-\alpha+\alpha r)}{(r+1)(1-r)}$$

At the first stage, the upstream monopolist maximizes total profit, the first-order condition (FOC) is:

$$\frac{\partial\left[u\left(\frac{1}{r^2-1}(r + P_i - rP_j - 1) + \frac{1}{r^2-1}(r + P_j - rP_i - 1)\right)(1-2\alpha)\right]}{\partial u} = 0$$

From there the equilibrium input price:

$$u^{B} = \frac{1}{-2\alpha + 2\alpha r + 2}$$

By substituting the equilibrium input price into the expressions, we can determine the equilibrium outcomes:

$$q_i^{B} = \frac{1}{2(r+1)(2-r)}$$

$$P_i^{B} = \frac{1}{2-r}\left(\frac{1}{-2\alpha+2\alpha r+2} - r - \frac{\alpha}{-2\alpha+2\alpha r+2} + \frac{\alpha r}{-2\alpha+2\alpha r+2} + 1\right) = \frac{3-2r}{2(2-r)}$$

$$v_i^{B} = -\frac{1}{4}(r-1)\frac{-3\alpha+2\alpha r+1}{(r+1)(r-2)^2(-\alpha+\alpha r+1)}$$

$$v_j^{B} = \frac{1}{4}(1-r)\frac{-3\alpha+2\alpha r+1}{(r+1)(r-2)^2(-\alpha+\alpha r+1)}$$

$$v_u^{B*} = \frac{-1}{2(r+1)(-\alpha+\alpha r+1)(r-2)}$$

$$v_{IND}^{B*} = \frac{3-2r}{2(r-2)^2(r+1)}$$

$$N_i^{B*} = \frac{1}{4}(1-r)\frac{-3\alpha+2\alpha r+1}{(r+1)(r-2)^2(-\alpha+\alpha r+1)} + \alpha\left(\frac{-1}{2(r+1)(-\alpha+\alpha r+1)(r-2)}\right)$$
$$= \frac{(1-r+\alpha+3\alpha r-2\alpha r^2)}{4(\alpha r-\alpha+1)(r-2)^2(r+1)}$$

The utility function (U), with symmetry $q_i = q_j = q$:

$$U = q_i + q_j - \frac{q_i^2 + q_j^2 + 2rq_iq_j}{2} = 2q - q^2(1+r)$$

Consumer surplus function (Y) is given by:

$$Y = 2q - q^2(1+r) - 2pq$$

Substituting the equilibrium values it is possible to obtain the equilibrium measures of welfare:

$$CS^{B*} = 2\left(\frac{1}{2(r+1)(2-r)}\right) - \left(\frac{1}{2(r+1)(2-r)}\right)^2(1+r)$$
$$-2\left(\frac{1}{2(r+1)(2-r)}\right)\left(\frac{3-2r}{2(2-r)}\right)$$
$$= \frac{1}{4(r+1)(r-2)^2}$$

$$SW^{B*} = CS^{B*} + v_{IND}^{B*} = \frac{1}{4(r+1)(r-2)^2} + \frac{3-2r}{2(r-2)^2(r+1)} = \frac{7-4r}{4(r-2)^2(r+1)}$$

Auxiliary derivatives:

$$\frac{\partial \left(u^{B*} \right)}{\partial \alpha} = \frac{(1-r)}{2(\alpha r - \alpha + 1)^2} > 0$$

$$\frac{\partial \left(\frac{(1-r)}{2(\alpha r-\alpha+1)^2} \right)}{\partial r} = \frac{1}{2}\frac{\alpha r - \alpha - 1}{(\alpha r - \alpha + 1)^3} < 0$$

$$\frac{\partial P_i^{B*}}{\partial \alpha} = 0$$

$$\frac{\partial P_i^{B*}}{\partial r} = -\frac{1}{2(r-2)^2} < 0$$

$$\frac{\partial \left(v_i^{B*} \right)}{\partial \alpha} = \frac{(1-r)}{4(\alpha r - \alpha + 1)^2 (r-2)(r+1)} < 0$$

$$\frac{\partial \left(v_u^{B*} \right)}{\partial \alpha} = \frac{(1-r)}{2(\alpha r - \alpha + 1)^2 (2-r)(r+1)} > 0$$

$$\frac{\partial \left(v_{IND}^{B*} \right)}{\partial \alpha} = 0$$

$$\frac{\partial \left(N_i^{B*} \right)}{\partial \alpha} = \frac{1}{4(\alpha r - \alpha + 1)^2 (2-r)} > 0$$

$$\frac{\partial CS^{B*}}{\partial \alpha} = 0$$

$$\frac{\partial SW^{B*}}{\partial \alpha} = 0$$

Proof of Proposition 1 (PVBO):

$$u^{C*} - u^{B*} = \frac{1}{2(1-\alpha)} - \frac{1}{-2\alpha + 2\alpha r + 2} = \frac{r\alpha}{2(\alpha r - \alpha + 1)(1-\alpha)} > 0$$

$$\frac{\partial \left(u^{C*} - u^{B*} \right)}{\partial \alpha} = \frac{\partial \left(\frac{r\alpha}{2(\alpha r-\alpha+1)(1-\alpha)} \right)}{\partial \alpha} = \frac{(\alpha^2 r - \alpha^2 + 1) r}{2(\alpha r - \alpha + 1)^2 (\alpha - 1)^2} > 0$$

$$N_u = uQ(1-2\alpha)$$

$$\frac{\partial N_u}{\partial u} = Q(1-2\alpha) + u(1-2\alpha)\frac{\partial Q}{\partial u}$$

$$Q^C - Q^B = 2\frac{-u + u\alpha + 1}{r+2} - 2\frac{(u - u\alpha + u\alpha r - 1)}{(r-2)(r+1)} = 2\frac{(2u\alpha - r + ur) r}{(r+1)(r+2)(2-r)} \gtrless 0$$

$$\frac{6Q^C}{6u} - \frac{6Q^B}{6u} = \frac{6\left(2^{\frac{-m+m□+1}{r+2}}\right)}{6u} - \frac{6\left(2^{\frac{(m-m□+m□r-1)}{(r-2)(r+1)}}\right)}{6u}$$

$$= -\frac{2(1-\alpha)}{r+2} - \left(-\frac{2(\alpha r-\alpha+1)}{(2-r)(r+1)}\right) = \frac{(2\alpha+r)r}{(r+1)(r+2)(2-r)} > 0$$

$$o^C = -\frac{2(1-\alpha)}{r+2}\left(\frac{u}{2^{\frac{-m+m□+1}{r+2}}}\right) = -\frac{(1-\alpha)u}{u\alpha-u+1} < 0$$

$$o^B = -\frac{2(\alpha r-\alpha+1)}{(2-r)(r+1)}\left(\frac{u}{2^{\frac{(m-m□+m□r-1)}{(r-2)(r+1)}}}\right) = \frac{(\alpha r-\alpha+1)u}{u-u\alpha+u\alpha r-1} < 0$$

$$□o^C_m□ = u\frac{(1-\alpha)}{u\alpha-u+1}$$

$$□o^B_m□ = u\frac{\alpha-\alpha r-1}{u-u\alpha+u\alpha r-1}$$

$$\frac{6□o^C_m□}{6\alpha} = \frac{6\left(\frac{(1-□)m}{m□-m+1}\right)}{6\alpha} = -\frac{u}{(-u+u\alpha+1)^2} < 0$$

$$\frac{6□o^B_m□}{6\alpha} = \frac{6\left(u\frac{□-□r-1}{m-m□+m□r-1}\right)}{6\alpha} = -u\frac{1-r}{(u-u\alpha+u\alpha r-1)^2} < 0$$

$$□o^C_m□ - □o^B_m□ = \frac{(1-\alpha)u}{u\alpha-u+1} - u\frac{\alpha-\alpha r-1}{u-u\alpha+u\alpha r-1}$$

$$= -\frac{u\alpha r}{(1-u+u\alpha)(1-u+u\alpha(1-r))} < 0$$

$$u^C > u^B$$

Proof of Proposition 2 (PVBO):

$$v_i^C - v_i^B = \frac{1-3\alpha-\alpha r}{4(1-\alpha)(r+2)^2} - \frac{1}{4}(1-r)\frac{-3\alpha+2\alpha r+1}{(r+1)(r-2)^2(-\alpha+\alpha r+1)}$$

$$= \frac{1}{4}r\frac{-\alpha^2 r(1-r)(4-2r-r^2)-2\alpha(2-r)(r+r^2+2)+2r^2}{(r+1)(r-2)^2(r+2)^2(1-\alpha)(1-\alpha+\alpha r)} \ \square\ 0$$

$$v_u^C - v_u^B = \frac{1}{2(1-\alpha)(r+2)} - \frac{1}{2(r+1)(-\alpha+\alpha r+1)(r-2)}$$

$$= \frac{1}{2}r\frac{2\alpha-r+2\alpha r-\alpha r^2}{(2-r)(r+2)(r+1)(1-\alpha)(-\alpha+\alpha r+1)} \ \square\ 0$$

$$v_{IND}^C - v_{IND}^B = \frac{1}{2}r^2\frac{2r+r^2-4}{(r+1)(r-2)^2(r+2)^2} < 0$$

$$N_i^{C\square} - N_i^{B\square} = \frac{(\alpha+\alpha r+1)}{4(1-\alpha)(r+2)^2} - \frac{(1-r+\alpha+3\alpha r-2\alpha r^2)}{4(\alpha r-\alpha+1)(r-2)^2(r+1)}$$

$$= \frac{\left(16\alpha^2-8\alpha r-8\alpha+2r^2-2\alpha r^2+12\alpha^2 r+4\alpha r^3-6\alpha^2 r^2-5\alpha^2 r^3+\alpha^2 r^4\right) \square\ r}{4(\alpha r-\alpha+1)(1-\alpha)(r+2)^2(r-2)^2(r+1)} \ \square\ 0$$

Proof of why r being low, contributes to the profit reversal result $\left(v_i^C - v_i^B < 0\right)$, represented in Figure 1:

If $\alpha = 0$, then $u^C = u^B = \frac{1}{2}$, and Cournot is always more profitable than Bertrand:

$$v_i\left(u^C = \frac{1}{2}\right) - v_i\left(u^B = \frac{1}{2}\right) = \frac{1}{2}\frac{r^3}{(r+1)(r-2)^2(r+2)^2} > 0$$

As $\alpha > 0$, input price will be higher than $\frac{1}{2}$, due to the internalization of the upstream profit.

If the input price were the same under the two competition types $\left(u^{\square} = u^C = u^B\right)$, but assuming $\alpha > 0$, then Cournot would still be more profitable than Bertrand:

$$v_i\left(u^{C\square}\right) - v_i\left(u^{B\square}\right)$$

$$= \left(\frac{r+3}{2(r+2)}\frac{1}{2(r+2)} - \frac{u^{\square}}{2(r+2)}\right) - \left(\frac{3-2r}{2(2-r)}\frac{1}{2(r+1)(2-r)} - \frac{u^{\square}}{2(r+1)(2-r)}\right)$$

$$= \frac{1}{2}\frac{r^3}{(r+1)(r-2)^2(r+2)^2} + \frac{1}{4}r^2\frac{2u^{\square}-1}{(2-r)(r+2)(r+1)} > 0$$

However $u^C > u^B$ and that is what is driving the profit reversal result. Furthermore we also know that $u^{C\square} > u^{B\square} > \frac{1}{2}$, because $\alpha > 0$:

$$v_i\left(u^{C\square}\right) - v_i\left(u^{B\square}\right) = \left[\frac{r+3}{2(r+2)}\frac{1}{2(r+2)} - \left(\frac{1}{2} + \square_B + f\right)\frac{1}{2(r+2)}\right] - \left[\frac{3-2r}{2(2-r)}\frac{1}{2(r+1)(2-r)} - \left(\frac{1}{2} + \square_B\right)\frac{1}{2(r+1)(2-r)}\right]$$

$$= \frac{1}{2}\frac{r^3}{(r+1)(r-2)^2(r+2)^2} + \frac{1}{2}\frac{r^2\square_B}{(2-r)(r+2)(r+1)} - f\frac{1}{2(r+2)} \;\square\; 0$$

Where:

$$\square_B = u^{B\square} - u^B(\alpha = 0) = \frac{1}{2(1-\alpha(1-r))} - \frac{1}{2} = \frac{1}{2}\alpha\frac{1-r}{1-\alpha+\alpha r}$$

$$\frac{\partial\left(\frac{1}{2}\alpha\frac{1-r}{1-\alpha+\alpha r}\right)}{\partial r} = -\frac{1}{2}\frac{\alpha}{(\alpha r-\alpha+1)^2} < 0$$

$$f = u^{C\square} - u^{B\square} = \frac{1}{2(1-\alpha)} - \frac{1}{2(1-\alpha(1-r))} = \frac{1}{2}\alpha\frac{r}{(1-\alpha)(1-\alpha+\alpha r)}$$

$$\frac{\partial f}{\partial r} = \frac{\partial\left(\frac{1}{2}\alpha\frac{r}{(1-\alpha)(1-\alpha+\alpha r)}\right)}{\partial r} = \frac{1}{2}\frac{\alpha}{(1-\alpha+\alpha r)^2} > 0$$

$$CS^C - CS^B = \frac{(r+1)}{4(r+2)^2} - \frac{1}{4(r+1)(r-2)^2} = \frac{(r^2-2r-4)r^2}{4(r+2)^2(r-2)^2(r+1)} < 0$$

$$SW^C - SW^B = \frac{3r+7}{4(r+2)^2} - \frac{7-4r}{4(r-2)^2(r+1)} = \frac{(2r+3r^2-12)r^2}{4(r+2)^2(r-2)^2(r+1)} < 0$$

To explain welfare measures, let us see the difference in quantities and prices, between Cournot and Bertrand:

$$q_i^{C\square} - q_j^{B\square} = \frac{1}{2(r+2)} - \frac{1}{2(r+1)(2-r)} = \frac{r^2}{2(r+1)(r+2)(r-2)} < 0$$

$$P_i^{C\square} - P_j^{B\square} = \frac{r+3}{2(r+2)} - \frac{3-2r}{2(2-r)} = \frac{r^2}{2(r+2)(2-r)} > 0$$