The Amazing Geometry of Price Competition with Quality Dependent Production Costs

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Abstract

This paper provides a full characterization of price competition in a vertical differentiation model with quality dependent marginal production costs. For each vector of qualities, we determine the Nash equilibrium and show graphically the different market regions for different values of the lowest quality valuation parameter, which reveals an amazing geometry. Besides the classical cases of high-quality monopoly or duopoly with partial or full coverage, we show that, when the high-quality firm has a too high quality, the equilibrium is a low-quality monopoly. Moreover, for positive lowest quality valuation, there always exist interior and corner full coverage duopoly equilibria. On the contrary, partial coverage duopoly do not exist when the lowest consumers' valuation of quality is high whereas high-quality monopoly are not possible for low values of lowest quality valuation. Our findings are the backbone of future analysis of quality choices and may be relevant for firms and policy makers.

Keywords: Vertical differentiation; market coverage configurations; quality valuation dispersion; quality dependent marginal costs.

JEL classification: L13; L15; D43

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1 Introduction

This paper considers a price competition game in a vertically differentiated market. The two potential competitors sell products with different qualities under unit production costs that increase with quality. Depending on the parameters (lowest quality valuation, quality valuation dispersion and marginal cost coefficient) and the exogenous quality vectors, we discuss, in a quite general setup, the existence of different price Nash equilibria configurations.

Why revisit such an old issue? Do we have something new to say and, if yes, is it relevant? We do have something new to say, because, in the Vertical Product Differentiation (VPD) literature, the price game has never been fully characterized, namely by considering all possible market configurations (monopoly or duopoly, with partial or full coverage) and identifying the conditions that define each type of Nash equilibrium. Thus, one of our main contributions is precisely the complete characterization of the price game. Additionally, by being exhaustive, we uncover the existence of unexpected market regions, which have counterintuitive comparative statics, which is also an important contribution. Our analysis allows us to know, for a given combination of the parameters (lowest quality valuation, quality valuation dispersion and marginal cost coefficient) and a given vector of offered qualities, which is the type of market configuration that will hold in equilibrium. Knowing the type of equilibrium market configuration, for each combination of offered qualities is relevant both for firms, when taking quality decisions, as well as for policy makers, when evaluating the impact of policy decisions that may affect quality choices, such as imposing minimum quality standards.

Monopoly and duopoly market structures have been widely explored in the VPD literature. However, most literature does not study both scenarios simultaneously and does not provide a complete characterization of when each type of equilibrium market configuration holds. Moreover, frequently only partial coverage or full coverage solutions are explored. For instance, Tirole (1988), Lambertini (1996), Wang and Yang (2001) and Wang (2003) study duopoly full coverage scenario, whereas Choi and Shin (1992) and Motta (1993) focus on duopoly partial coverage scenario. Wauthy (1996) and Liao (2008) cover duopoly with full and partial coverage scenarios. In this paper, we discuss whether a monopoly or duopoly will arise and, in both cases, we also discuss whether the market is fully covered or not. Both issues are critical for consumers and, thus, for competition authorities. Whether the whole market is served or not has a great impact on consumers' welfare: if some consumers are not served, we may argue that the market outcome is not promoting economic and social inclusion. Under VPD, consumers have different preferences regarding quality and, hence, different willingness to pay for it. Some consumers are eager to have the highest quality product or the last innovation and are willing to pay for it, whereas others do not bother having a lower quality good, but pay less. It may not be in the competitors' interest to satisfy all consumers, that is, to give them positive utility when they buy the product. If this is the case, then the market will not be fully covered and some consumers are excluded from buying. This may happen, for instance, when not the whole population has access to housing, to buy a car, to buy a new mobile phone model, to buy a holiday package, etc.

It should be noted that our assumption of quality dependent costs is a very realist assumption and still underexplored in the VPD literature, which further contributes to the relevance of our findings. VPD literature can be categorized according to the assumption that is made about the nature of the costs of quality improvement. Some papers assume that production costs do not depend on the quality (like Tirole 1988, Choi and Shin 1992, or Wauthy 1996). Others assume that, in order to improve quality, firms have to incur investment costs, such as advertising and research and development (like Shaked and Sutton1982, Bonanno 1986, or Lambertini and Tampieri 2012). This type of costs has been denoted by fixed quality costs (Motta, 1993), since they are normally incurred before production takes place and are sunk costs by the time production occurs. However, improving quality can also have implications on production costs and, in particular, on the marginal production costs. In fact, higher quality may require more expensive inputs or a more specialized labour force, which implies increasing marginal costs. Increasing unit costs have been assumed by Mussa and Rosen (1978), Aoki and Prusa (1996), Lehman-Grube (1997), Lambertini (1999), Schmidt (2006) and Lambertini and Tedeschi (2007), among others.

Our price competition game is essential to understand simultaneous or sequential quality choices in a dynamic quality-price game with VPD and quality dependent marginal costs. The price equilibrium market configurations are needed to determine the subgame perfect Nash equilibria, since, when checking for deviations from equilibrium candidates' qualities, one must consider all possible deviations and, hence, we need to consider all the possible pricing Nash equilibria configurations. Therefore, our analysis is a backbone for future analysis of dynamic games with quality choices.

Some may argue that we only need to explore market configurations that hold for quality combinations that can be chosen by firms in the equilibrium of a dynamic quality-price game. However, this reasoning is flawed by several reasons. The first is that we can only know which qualities are chosen in equilibrium in a dynamic game if we analyze all possible quality combinations, as argued before. Moreover, the argument is implicitly based on the assumption that firms can deterministically choose their quality levels. This may not be true. It may be more realistic to assume that firms invest in R&D and the quality of their product is randomly determined as a function of the R&D investment. Under this assumption, the observed qualities may be quite different from the ones firms would choose in a deterministic world. Lastly, the quality choices may be conditioned by regulation, which may «force» firms to make choices they would not do in an unregulated context and may even influence the market configuration that will hold.

Our results show that, depending on the parameter values and the combinations of quality levels, many types of equilibria configurations may exist, besides the expected cases of high-quality monopoly or duopoly with partial or full coverage. A very interesting result is that, with quality dependent marginal costs, for high quality levels of the high-quality firm, the market configuration is a low-quality monopoly. The reason for this result is that the high-quality firm is trapped by offering a too high quality that is too expensive to produce and, hence, is unable to offer a competitive surplus to the consumer. Our analysis also reveals the existence of a duopoly configuration with a corner solution, where the market is fully covered but the lowest quality valuation consumer gets a nil surplus. This type of equilibrium can be seen as an intermediate solution between the partial and the interior full coverage Nash equilibria. Wauthy (1996) also identified this type of solution, for the case of costless quality. Moreover, we also show that there are parameter configurations such that neither a duopoly nor an unconstrained monopoly solution hold. For such intermediate cases, the unique Nash equilibrium is a constrained monopoly where the firm which has a «surplus advantage» is able to behave as a monopolist by charging a price that offers to consumers the same surplus that is offered by the competitor when charging a price equal to its marginal cost. In the case of the low-quality constrained full coverage monopoly, there are unexpected comparative statics as the monopolist price is decreasing with the quality valuation dispersion and with the level of quality differentiation among the two firms, and profits are increasing with the marginal cost coefficient and decreasing with the quality valuation dispersion and with the level of quality differentiation.

Which type of market configuration holds and how large is each market region depends on the lowest consumers' valuation of quality, the quality valuation dispersion and the marginal cost coefficient. For instance, when the lowest quality valuation is low, the predominant configurations are the ones with partial market coverage and a high-quality monopoly cannot exist. On the contrary, for high lowest quality valuations, a duopoly with partial coverage is no longer possible and both high-quality and low-quality monopolies are possible. Our results show that decreasing the marginal cost coefficient and/or increasing the dispersion of the quality valuation enlarge the set of possible quality combinations. Moreover, increasing the quality valuation dispersion implies that duopoly configurations hold for a larger set of values of the lowest valuation parameter.

The article is organized as follows. In the next section we describe the model and preliminary results. In section 3, we study the price Nash equilibria, obtaining analytically the equilibrium prices for different market configurations and identifying the conditions for each type of Nash equilibrium to hold. Section 4 explores the geometry of the different market configurations for different values of the lowest quality valuation parameter. Finally, section 5 discusses the main implications of our results and summarizes the conclusions. An Appendix contains all proofs.

2 The model and some preliminary results

2.1 The model

We consider a standard model of VPD (Mussa and Rosen 1978, Gabszewicz and Thisse 1979, Shaked and Sutton 1982, and Tirole 1988). There are two firms indexed by $i = 1, 2$. The consumer either buys one unit of the product or none. If the consumer does not buy his utility is nil, whereas if he buys a product from firm i, his utility is given by:

$$U(\theta) = \theta k_i - p_i,$$

where k_i represents the quality of the product sold by firm i and p_i is the corresponding price. The parameter θ is a taste parameter that reflects how much the consumer values quality - consumer's quality valuation. This parameter is uniformly distributed across the population between $\underline{\theta} > 0$, the lowest quality valuation, and $\overline{\theta} = \underline{\theta} + d$, the highest quality valuation. Therefore, $\theta \in [\underline{\theta}, \underline{\theta} + d]$, with density $f(\theta) = \frac{1}{d}$, where d is the quality valuation (upward) dispersion.[1]

Firms face constant marginal production costs that depend in a quadratic way on quality. That is, total production costs are given by:

$$C(q_i) = c_i \cdot q_i = ck_i^2 \cdot q_i$$

where q_i is the quantity produced and c is marginal cost coefficient.[2]

[1] The fact that we fix $\underline{\theta}$ and define $d = \overline{\theta} - \underline{\theta}$ has implications for comparative statics exercises as increasing d means having higher valuation consumers.

[2] In the derivation of the equilibrium prices, it is easy to see that the Nash equilibrium prices could be derived for generic marginal costs that are increasing with quality. However to derive the limits of each market region, the functional form has to be specified. We assume that marginal costs vary quadratically with quality, because

In our model we assume that the quality of each firm's product is exogenous, and for each quality vector (k_1, k_2), the firms simultaneously decide their prices. Without loss of generality, we assume that $k_2 > k_1$ (with $k_1 > k_2$, we would have similar equilibria, but with the roles of the two firms reversed).[3] Hereafter, firm 2's product is high-quality, whereas firm 1's product is low-quality.

2.2 Some important cutoff valuations and preliminary results

The consumer's choice between buying from firm i or not buying depends on whether buying from firm i gives the consumer a positive net utility or not:

$$U_i(\theta) = \theta k_i - p_i \geq 0.$$

Let $\widehat{\theta}_i(p_i) = \frac{p_i}{k_i}$ be the quality valuation of the consumer for which the previous condition is satisfied in equality. Thus, the consumer with valuation $\widehat{\theta}_i$ is indifferent between buying from firm i or not buying at all. All consumers with $\theta > \widehat{\theta}_i$ strictly prefer to buy from firm i than not to buy, whereas all consumers with $\theta < \widehat{\theta}_i$ prefer not to buy than to buy from firm i.

Note that the utility function is increasing with θ. This has implications on the way consumers choose between the two firms. The following results are easy to show:

Lemma 1 *If the highest valuation consumer, $\underline{\theta} + d$, prefers the low-quality product than the high-quality good, then all the consumers prefer the low-quality product.*

Proof. See Appendix. ■

A similar result holds when the lowest valuation consumer prefers the high-quality product:

Lemma 2 *If the lowest valuation consumer, $\underline{\theta}$, prefers the high-quality product than the low-quality good, then all the consumers prefer the high-quality product.*

Proof. See Appendix. ■

The proofs show that, for a given quality differential, the consumer's decision depends on the prices differential, $p_2 - p_1$. For both firms to have positive demand, the price differential must be intermediate. The price differential cannot be too high, since otherwise all consumers would prefer the low-quality product. On the other hand, the price differential cannot be too low, otherwise all consumers would prefer the high-quality product.

this is the most common assumption. In the paper, to denote the marginal costs of firm i, we use interchangeably c_i and ck_i^2.

[3] If $k_2 = k_1$, this would resort to an homogenous product Bertrand analysis.

Lemma 3 *If both firms have positive demand, the high-quality firm serves the higher valuation consumers, whereas the low-quality firm serves the lower valuation consumers.*

Proof. See Appendix. ■

As $k_2 > k_1$, consumers prefer to buy from firm 1 than from firm 2 if:

$$\theta k_1 - p_1 \geq \theta k_2 - p_2 \quad \Leftrightarrow \quad \theta \leq \frac{p_2 - p_1}{k_2 - k_1}$$

Let $\widetilde{\theta} = \frac{p_2 - p_1}{k_2 - k_1}$ be the value of θ such that the previous expression holds in equality. In other words, $\widetilde{\theta}$ is the indifferent consumer between buying from firm 1 or buying from firm 2. The consumers with $\theta > \widetilde{\theta}$ strictly prefer to buy from firm 2, whereas the consumers with $\theta < \widetilde{\theta}$ strictly prefer to buy from firm 1.

3 Nash equilibrium of the pricing game

If $k_2 > \frac{\overline{\theta} + d}{c}$, no consumer will ever buy from firm 2 even if it charges a price equal to its marginal cost. Firm 2 is offering a too high quality, which is too costly and thus no consumer is interested in buying it. Hence, if $k_2 > \frac{\overline{\theta} + d}{c}$ firm 1 has a guaranteed monopoly. Thus, we only study the Nash equilibrium of the price game when $k_2 \leq \frac{\overline{\theta} + d}{c}$, as we already know the market configuration for $k_2 > \frac{\overline{\theta} + d}{c}$.

3.1 Monopoly Configurations

Our analysis starts with the monopoly configurations. It makes sense to analyze these cases first because whether a monopoly or duopoly solution holds depends on whether it is profitable for the firm with a "surplus advantage" to charge a low enough price to be a monopolist or if it is preferable to charge a higher price and share the market with the other firm.[4]

A crucial message in this section is that we cannot study only the case where the firm with "surplus advantage" can behave as an unconstrained monopolist. The firm can only do that when the surplus advantage is high. In the other cases, the surplus constraint will be binding.[5]

[4] We say that a firm has a "surplus advantage" if, with cost marginal pricing, all the consumers prefer to buy the product of that firm than the product of the competitor.

[5] The way we formalize our problem is such that the constrained monopoly solution seems a very natural solution. However, if one starts with the unconstrained optimization problem and do not fully explore the limits of each market region, one can easily miss identifying this type of Nash equilibrium of the price game.

3.1.1 High-quality monopoly

If the quality of firm 1 is very low, it may happen that firm 2 is able to profitably offer to all consumers who buy the product an higher surplus than the one offered by firm 1 at $p_1 = c_1$. That is, $U_2(\theta) > \overline{U}_1(\theta)$ for all consumers who buy a product (Figure 1 illustrates the scenario), where $\overline{U}_1(\theta) = \theta k_1 - ck_1^2$ is the surplus offered by firm 1 with marginal cost pricing, $p_i = ck_i^2$, which is the maximal surplus that firm 1 can profitably offer.

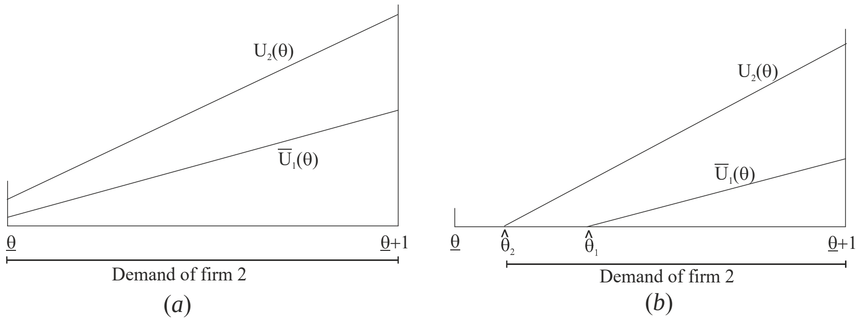

Figure 1: The high-quality firm can offer a higher surplus than the maximal surplus the low-quality firm can offer: (a) full coverage; (b) partial coverage.

To be a monopolist, firm 2 maximizes its profit subject to the constraint that the lowest valuation consumer who buys a product gets a surplus at least as high, if he buys from firm 2, as if he buys a product from firm 1 at price $p_1 = c_1$.[6] This "surplus constraint" can be written as follows:[7]

$$\max\left[\underline{\theta}, \widehat{\theta}_1(c_1)\right] k_2 - p_2 \geq \max\left[\underline{\theta}, \widehat{\theta}_1(c_1)\right] k_1 - c_1 \quad \Leftrightarrow \quad p_2 \leq \max\left[\underline{\theta}, \widehat{\theta}_1(c_1)\right](k_2 - k_1) + c_1$$

If the lowest quality valuation consumer, $\underline{\theta}$, gets a non-negative surplus if he buys from firm 1 at $p_1 = c_1$ (that is, $\widehat{\theta}_1(c_1) \leq \underline{\theta}$, or equivalently, $k_1 \leq \frac{\underline{\theta}}{c}$), then, to be a monopolist, firm 2 must cover the whole market and offer consumer $\underline{\theta}$ at least the same surplus he would get from firm 1. Hence, firm 2 solves the following problem:

$$\max_{p_2} \Pi_2 = (p_2 - c_2) \quad \text{subject to} \quad p_2 \leq \underline{\theta}\,(k_2 - k_1) + c_1$$

On the other hand, if $\widehat{\theta}_1(c_1) > \underline{\theta}$ or, equivalently, if $k_1 > \frac{\underline{\theta}}{c}$, the "surplus constraint" is given

[6] We need only to consider the surplus constraint at $p_1 = c_1$, because there cannot exist a Nash equilibrium where firm 2 is a monopolist with a binding surplus constraint and $p_1 > c_1$, as firm 1 would gain to a slightly lower price (it would get a positive demand and a positive margin).

[7] Recall that $\widehat{\theta}_1(p_1) = \frac{p_1}{k_1}$, hence $\widehat{\theta}_1(c_1) = = \frac{c_1}{k_1} = ck_1$.

by:

$$\widehat{\theta}_1(c_1)k_2 - p_2 \geq \widehat{\theta}_1(c_1)k_1 - c_1 \quad \Leftrightarrow \quad p_2 \leq ck_1k_2$$

In this case, firm 2 may or may not cover the whole market and solves:

$$\max_{p_2} \Pi_2 = \left(\underline{\theta} + d - \frac{p_2}{k_2}\right)(p_2 - c_2) \quad \text{subject to} \quad \underline{\theta}k_2 \leq p_2 \leq ck_1k_2$$

For high $\underline{\theta}$ and relatively small qualities, the Nash equilibrium in the price stage game corresponds to a high-quality constrained monopoly:

Proposition 1 *Assuming $k_1 < k_2 \leq \frac{1}{c}(\underline{\theta} + d)$, if $k_1 + k_2 < \frac{1}{c}(\underline{\theta} - d)$, firm 2 can behave as a high-quality constrained monopolist and covers the whole market. The equilibrium prices are*

$$p_1^* = ck_1^2 \quad \textit{and} \quad p_2^* = \underline{\theta}(k_2 - k_1) + ck_1^2$$

and the corresponding equilibrium profits are

$$\Pi_1^* = 0 \quad \textit{and} \quad \Pi_2^* = (k_2 - k_1)(\underline{\theta} - c(k_2 + k_1))$$

There is no Nash equilibrium with a partial coverage high-quality monopoly.

Proof. See Appendix. ■

The equilibrium price of the high-quality monopolist is increasing with the marginal cost of the potential competitor, ck_1^2, the lowest quality valuation, $\underline{\theta}$, and with the quality differential, $k_2 - k_1$. The monopolist profit is also increasing with $\underline{\theta}$ and with $(k_2 - k_1)$, but decreasing with c. However, neither the price nor the profit depend on d, which is a bit surprising. The reason for this is that the "surplus constraint" is binding in equilibrium and the impact of a change in the parameters on price and profit comes through its impact on the "surplus constraint". Increasing $\underline{\theta}$ and c relaxes the "surplus constraint", and thus allows firm 2 to charge a higher prices and get higher profit. Since the "surplus constraint" only depends on the valuation of the lowest valuation consumer, it is not affected by the dispersion coefficient, d, which explains why this parameter does not influence the high-quality monopoly profit.

It should be highlighted that the high-quality monopoly can only hold for $\underline{\theta} > d$. Thus, a higher dispersion in consumers' preferences implies that the high-quality monopoly will only hold for higher values of $\underline{\theta}$.

3.1.2 Low-quality firm monopoly

In our model, the maximal surplus that a firm can profitably offer to consumer of type θ, $\overline{U_i} = \theta k_i - ck_i^2$, is not linear in the quality offered, as it happens with nil or symmetric quality independent marginal costs. This has a strong implication in our results. As marginal costs of production are a quadratic function of quality, the high-quality firm is "trapped" by offering a very high-quality, as its marginal production costs are so high that the surplus it offers to consumers is low. Hence the low-quality firm may have a "surplus advantage" and, consequently, may be able to be a monopolist.

We know that, if the highest valuation consumer prefers to buy from firm 1, everyone else also prefers to buy from firm 1. If firm 2 has a quality level close to but smaller than $k_2 = \frac{\underline{\theta}+d}{c}$, firm 1 may be able to profitably offer a higher surplus to the highest valuation consumer and be a monopolist. Figure 2 illustrates a scenario where firm 1 has a "surplus advantage". To be a monopolist, firm 1 is constrained to offer at least the surplus offered by firm 2 when pricing at marginal cost ($p_2 = ck_2^2$).

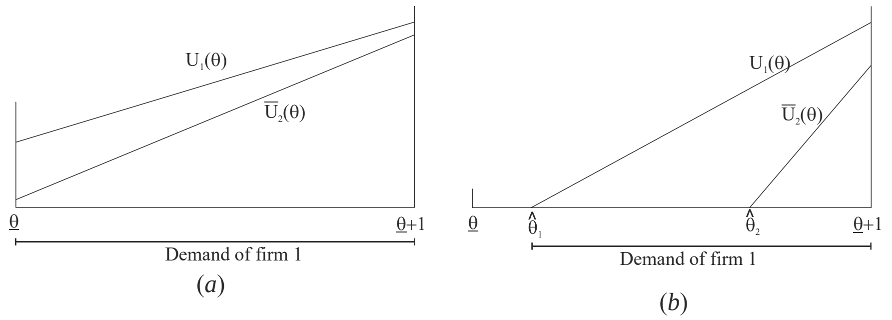

Figure 2: The low-quality firm can offer a higher surplus than the maximal surplus the high-quality firm can offer: (a) full coverage; (b) partial coverage.

If firm 1 wants to ensure a monopoly, it maximizes its profit subject to the constraint that the highest valuation consumer gets a higher surplus if he buys from firm 1 than if he buys from firm 2 at $p_2 = c_2$. In other words, the "surplus constraint" is:

$$(\underline{\theta} + d)k_1 - p_1 \geq (\underline{\theta} + d)k_2 - c_2 \quad \Leftrightarrow \quad p_1 \leq c_2 - (\underline{\theta} + d)(k_2 - k_1).$$

If $c_2 - (\underline{\theta} + d)(k_2 - k_1) \leq \underline{\theta}k_1$, the lowest valuation consumer gets a non-negative surplus and thus the whole market is covered. Firm 1 solves the following problem:

$$\max_{p_1} \Pi_1 = (p_1 - c_1) \quad \text{subject to} \quad p_1 \leq c_2 - (\underline{\theta} + d)(k_2 - k_1).$$

If $c_2 - (\underline{\theta} + d)(k_2 - k_1) > \underline{\theta} k_1$, firm 1 may or may not cover the whole market and solves the following problem:

$$\max_{p_1} \Pi_1 = \frac{1}{d}\left(\underline{\theta} + d - \frac{p_1}{k_1}\right)(p_1 - c_1) \quad \text{subject to} \quad \underline{\theta} k_1 \leq p_1 \leq c_2 - (\underline{\theta} + d)(k_2 - k_1).$$

For given $\underline{\theta}$, there are some quality vectors such that the solution of firm 1's maximization problem is not influenced by the "surplus constraint" and hence the Nash equilibrium corresponds to a low-quality unconstrained monopoly:

Proposition 2 *Assuming* $k_1 < k_2 \leq \frac{1}{c}(\underline{\theta} + d)$, *if* $k_1 > \frac{1}{c}(\underline{\theta} - d)$ *and* $k_2 < \frac{(\underline{\theta}+d) - \sqrt{(\underline{\theta}+d)^2 - 2c\left((\underline{\theta}+d)k_1 - ck_1^2\right)}}{2c}$ *or* $k_2 > \frac{(\underline{\theta}+d) + \sqrt{(\underline{\theta}+d)^2 - 2c\left((\underline{\theta}+d)k_1 - ck_1^2\right)}}{2c}$, *firm 1 behaves as a low-quality unconstrained monopoly and there is partial coverage. The Nash equilibrium prices are given by:*

$$p_1^* = \frac{ck_1^2 + k_1(\underline{\theta} + d)}{2} \quad \textit{and} \quad p_2^* = ck_2^2$$

and the corresponding equilibrium profits are

$$\Pi_1^* = \frac{k_1\left(\underline{\theta} + d - ck_1\right)^2}{4d} \quad \textit{and} \quad \Pi_2^* = 0$$

If $k_1 \leq \frac{1}{c}(\underline{\theta} - d)$ *and* $k_2 < \frac{(\underline{\theta}+d) - \sqrt{(\underline{\theta}+d)^2 - 4cdk_1}}{2c}$ *or* $k_2 > \frac{(\underline{\theta}+d) + \sqrt{(\underline{\theta}+d)^2 - 4cdk_1}}{2c}$, *firm 1 behaves as a low-quality unconstrained monopoly and there is full coverage. The Nash equilibrium prices are given by:*

$$p_1^* = \underline{\theta} k_1 \quad \textit{and} \quad p_2^* = ck_2^2$$

and the corresponding equilibrium profits are:

$$\Pi_1^* = k_1\left(\underline{\theta} - ck_1\right) \quad \textit{and} \quad \Pi_2^* = 0.$$

Proof. See Appendix. ■

For a low-quality unconstrained monopoly, the equilibrium price is increasing with its quality. It increases linearly when the market is fully covered because the price is such that the lowest valuation consumer gets zero surplus and, hence, does not depend on the marginal costs. It increases at an increasing rate if the market is only partially covered due to the shape of marginal costs. When the market is partially covered, the equilibrium and profits are increasing with the monopolist's quality level and with the dispersion parameter, d. These results are very intuitive, as increasing d implies higher valuation consumers and a large fraction of the market being served. However, if the market is fully covered, d no longer influences the prices and the profit because the monopolist is extracting all the surplus from the lowest valuation consumer.

However, there are also circumstances in which the "surplus constraint" is binding and the Nash equilibrium corresponds to a low-quality constrained monopoly:

Proposition 3 *Assuming* $k_1 < k_2 \leq \frac{1}{c}\left(\underline{\theta}+d\right)$, *if* $\frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c} \leq k_2 \leq \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c}$ *and* $k_1+k_2 > \frac{1}{c}\left(\underline{\theta}+2d\right)$, *firm 1 is a low-quality constrained full coverage monopoly and the Nash equilibrium profits are:*

$$\Pi_1^* = (k_2-k_1)\left[c\left(k_2+k_1\right)-\left(\underline{\theta}+d\right)\right] \quad \text{and} \quad \Pi_2^*=0$$

If $k_1 > \frac{\underline{\theta}-d}{c}$ *and* $\frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c} \leq k_2 \leq \frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c}$ *or* $\frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c}$ $\leq k_2 \leq \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c}$ *and* $k_2 > \frac{1}{c}\left(\underline{\theta}+d\right)-\frac{1}{2}k_1$, *firm 1 is a low-quality constrained partial coverage monopoly and the Nash equilibrium profits are*

$$\Pi_1^* = \frac{1}{2k_1}k_2\left(\frac{\underline{\theta}+d}{c}-k_2\right)(k_2-k_1)\left[c\left(k_2+k_1\right)-\left(\underline{\theta}+d\right)\right] \quad \text{and} \quad \Pi_2^*=0$$

In both cases, the Nash equilibrium prices are given by:

$$p_1^* = ck_2^2-\left(\underline{\theta}+d\right)(k_2-k_1) \quad \text{and} \quad p_2^*=ck_2^2$$

Proof. See Appendix. ■

The constrained monopoly can be interpreted as a intermediate solution between the unconstrained monopoly and the duopoly. Firm 1 is in a position such that, looking at the duopoly problem, it would like to decrease the price so as to increase demand. However that is not possible as everyone is already buying from firm 1. Looking at the interior monopoly problem, firm 1 would like to increase the price, but then it would no longer be a monopolist. Thus, the only possible solution for firm 1's optimization problem is a solution where the "surplus constraint" is binding, that is, $p_1^* = ck_2^2-\left(\underline{\theta}+d\right)(k_2-k_1)$.

The constrained monopolist price is increasing with the marginal cost of the competitor, hence it is increasing with c. However, p_1 is decreasing with $\underline{\theta}$, d and the quality differentiation (k_2-k_1), which is precisely the opposite of what one would expect. This counterintituive result is due to the fact that the constraint is binding in equilibrium. Therefore, the impact of a parameter on the solution comes through its impact on the constraint. The higher are $\underline{\theta}$, d and (k_2-k_1), the more difficult it will be for firm 1 to match the surplus offered by firm 2 and it will have to lower its price to continue being a monopolist. This also explains why the low-quality monopoly profit is decreasing with $\underline{\theta}$ and d.

3.2 Duopoly configurations

In this subsection, we consider the cases where both firms operate. First, we characterize the most common cases of full market coverage and partial market coverage. Next, we analyze the intermediate case of full market coverage where the profit maximizing price of the low-quality firm is a corner solution.

3.2.1 Full coverage duopoly

When both firms operate in the market with full coverage, firm 2 serves the consumers with higher valuation, while firm 1 supplies the consumers with lower valuation. Demands are given by (see Figure 3 (a)):

$$D_1 = \frac{1}{d}\left(\frac{p_2 - p_1}{k_2 - k_1} - \underline{\theta}\right) \quad \text{and} \quad D_2 = \frac{1}{d}\left(\underline{\theta} + d - \frac{p_2 - p_1}{k_2 - k_1}\right)$$

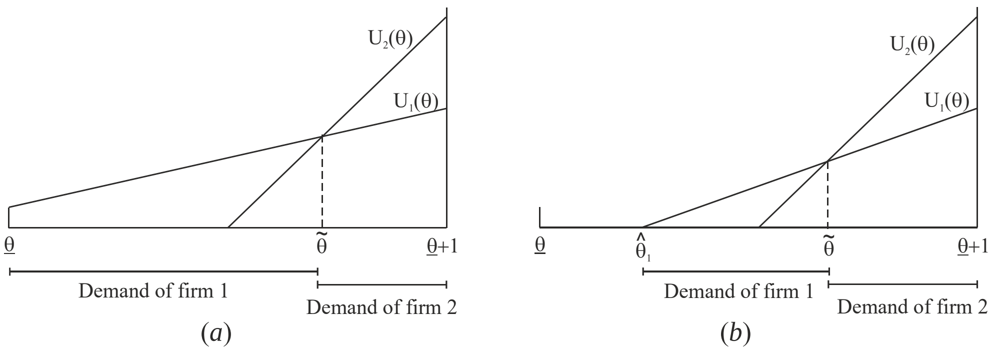

Figure 3: Demands under duopoly: (a) full coverage; (b) partial coverage.

The profit functions for the two firms are given by the following expressions:

$$\begin{aligned} \Pi_1 &= \frac{1}{d}\left(\frac{p_2 - p_1}{k_2 - k_1} - \underline{\theta}\right)(p_1 - c_1) \quad \text{s.t.} \quad p_1 \leq \underline{\theta} k_1 \\ \Pi_2 &= \frac{1}{d}\left(\underline{\theta} + d - \frac{p_2 - p_1}{k_2 - k_1}\right)(p_2 - c_2) \end{aligned}$$

Note that if $p_1 \leq \underline{\theta} k_1$ does not hold, the lowest valuation consumer would have a negative surplus and, hence, the market would not be fully covered and the profit function of firm 1 would be different. Assuming that the market is fully covered, the optimal price for firm 1 may be an interior solution, where $\frac{\partial \Pi_1}{\partial p_1}$ is nil, or a corner solution where the constraint is binding, $p_1 = \underline{\theta} k_1$.

If the solution of firm 1's maximization problem is an interior solution, the Nash equilibrium in prices is given by:

Proposition 4 *Assuming* $k_1 < k_2 \leq \frac{1}{c}\left(\underline{\theta} + d\right)$, *if* $\frac{k_1}{3}\left(2\underline{\theta} + d - 2ck_1\right) + \frac{k_2}{3}\left(\underline{\theta} - d - ck_2\right) \geq 0$ *and* $\frac{1}{c}\left(\underline{\theta} - d\right) \leq k_2 + k_1 \leq \frac{1}{c}\left(\underline{\theta} + 2d\right)$, *both firms operate in equilibrium and there is full coverage with an interior solution. The equilibrium prices are:*

$$
\begin{aligned}
p_1^* &= \frac{\left(d - \underline{\theta}\right)\left(k_2 - k_1\right) + 2ck_1^2 + ck_2^2}{3} \\
p_2^* &= \frac{(\underline{\theta} + 2d)(k_2 - k_1) + 2ck_2^2 + ck_1^2}{3}
\end{aligned}
$$

and the corresponding profits are:

$$
\begin{aligned}
\Pi_1^*\left(k_1, k_2\right) &= \frac{1}{9d}\left(k_2 - k_1\right)\left(d - \underline{\theta} + c(k_1 + k_2)\right)^2 \\
\Pi_2^*\left(k_1, k_2\right) &= \frac{1}{9d}\left(k_2 - k_1\right)\left(\underline{\theta} + 2d - c(k_1 + k_2)\right)^2
\end{aligned}
$$

Proof. See Appendix. ■

The equilibrium prices are increasing with c and d and depend on the quality differential $(k_2 - k_1)$. The fact that the marginal costs are a quadratic function of quality is reflected in the way the equilibrium prices depend on the qualities.

The impact of changes in the firm's product quality on its own price is clearly positive for the high-quality firm, since, when k_2 increases, the quality differential increases and the marginal costs also increase. For the low-quality firm, the sign is positive if $\underline{\theta} \geq d$, but it may be negative otherwise. When $\underline{\theta} < d$, increasing k_1 implies a reduction in the quality differential, hence tougher competition. However, it also leads to higher marginal costs. Which effect dominates depends on k_1. For low k_1, specifically for $k_1 < \frac{d - \underline{\theta}}{4c}$, p_1^* is decreasing with k_1 because the effect of the reduction on the quality differential is stronger than the impact of the increase in the marginal costs.

It is interesting to note that, for this equilibrium to hold, the sum of the qualities cannot be too low or too high relatively to $\frac{\underline{\theta}}{c}$. Moreover, the higher is the quality valuation dispersion, d, the larger is the set of vectors (k_1, k_2) such that an interior full coverage duopoly holds. In addition, if $\frac{k_1}{3}\left(2\underline{\theta} + d - 2ck_1\right) + \frac{k_2}{3}\left(\underline{\theta} - d - ck_2\right) < 0$, the Nash equilibrium cannot be a full coverage interior solution.

3.2.2 Partial coverage duopoly

If the equilibrium involves a duopoly with partial market coverage, firm 1 solves the following problem:[8]

$$\Pi_1 = \frac{1}{d}\left(\frac{p_2 - p_1}{k_2 - k_1} - \frac{p_1}{k_1}\right)(p_1 - c_1) \quad \text{s.t.} \quad p_1 > \underline{\theta} k_1$$

Note that if $p_1 > \underline{\theta} k_1$ does not hold, the lowest valuation consumer would have a non-negative surplus and hence the market would be fully covered and the profit function of firm 1 would no longer be given by the previous Π_1 expression. If the constraint is not binding, the Nash equilibrium is given by:

Proposition 5 *Assuming* $k_1 < k_2 \leq \frac{1}{c}(\underline{\theta} + d)$, *if* $k_2 \leq \frac{1}{c}(\underline{\theta} + d) - \frac{1}{2}k_1$ *and* $d(k_2 - k_1) - 3\underline{\theta} k_2 + ck_2(2k_1 + k_2) > 0$, *both firms operate and there is partial coverage. The equilibrium prices are:*

$$\begin{aligned} p_1^* &= \frac{k_1(\underline{\theta} + d)(k_2 - k_1) + 2ck_1^2 k_2 + ck_2^2 k_1}{4k_2 - k_1} \\ p_2^* &= \frac{2k_2(\underline{\theta} + d)(k_2 - k_1) + ck_1^2 k_2 + 2ck_2^3}{4k_2 - k_1} \end{aligned}$$

and the corresponding profits are:

$$\begin{aligned} \Pi_1^*(k_1, k_2) &= \frac{k_1 k_2 (k_2 - k_1)(\underline{\theta} + d + c(k_2 - k_1))^2}{d(4k_2 - k_1)^2} \\ \Pi_2^*(k_1, k_2) &= \frac{k_2^2 (k_2 - k_1)(c(2k_2 + k_1) - 2(\underline{\theta} + d))^2}{d(4k_2 - k_1)^2} \end{aligned}$$

Proof. See Appendix. ■

The equilibrium prices are both increasing with c, $\underline{\theta}$, d and $(k_2 - k_1)$, but the price of firm 2 is more sensitive to $\underline{\theta}$, d and $(k_2 - k_1)$ than the price of firm 1. This is because, when firm 1 increases its price, it looses consumers to firm 2, but it also looses the consumers who stop buying the product. So, firm 1 has a more elastic demand and, thus, it has a lower incentive than firm 2 to increase its price when $\underline{\theta}$, d and $(k_2 - k_1)$ increase.

3.2.3 Full coverage corner duopoly

The partial coverage and the interior full coverage solutions are mutually exclusive, but for certain quality vectors none of them holds. If $\frac{k_1}{3}(2\underline{\theta} + 1 - k_1) + \frac{k_2}{6}(2\underline{\theta} - 2 - k_2) < 0$ and $d(k_2 - k_1) - 3\underline{\theta} k_2 + ck_2(2k_1 + k_2) \leq 0$, the Nash equilibrium is neither an interior full coverage

[8] Figure 3 (b) shows the demands under partial coverage. The profit function of firm 2 is the same than in the full coverage duopoly case.

solution nor a partial coverage solution. In this case, the constraint in firm 1's full coverage optimization problem is binding. At $p_1 = \underline{\theta} k_1$, the profit of firm 1 is non-differentiable (for p_1 slightly below $\underline{\theta} k_1$, we are in the full coverage case, but $\frac{\partial \Pi_1}{\partial p_1} > 0$ so the firm would like to increase prices, but then the full coverage would no longer hold; for p_1 slightly above $\underline{\theta} k_1$, partial coverage holds, but $\frac{\partial \Pi_1}{\partial p_1} < 0$, so the firm would want to decrease prices, but this would imply that partial coverage no longer holds). Therefore, in this case, firm 1 is maximizing profit at $p_1 = \underline{\theta} k_1$.

The Nash equilibrium in this case is given by:

Proposition 6 *Assuming* $k_1 < k_2 \leq \frac{1}{c}\left(\underline{\theta} + d\right)$, *if* $\frac{k_1}{3}\left(2\underline{\theta} + 1 - k_1\right) + \frac{k_2}{6}\left(2\underline{\theta} - 2 - k_2\right) < 0$, $d\left(k_2 - k_1\right) - 3\underline{\theta} k_2 + c k_2 \left(2k_1 + k_2\right) \leq 0$ *and* $\underline{\theta} - \frac{d}{2} \leq \frac{\underline{\theta}(k_2 - 2k_1) + ck_2^2}{2(k_2 - k_1)} \leq \underline{\theta} + \frac{d}{2}$, *both firms operate and there is full coverage with a corner solution. The equilibrium prices are:*

$$
\begin{aligned}
p_1^* &= \underline{\theta} k_1 \\
p_2^* &= \frac{\left(\underline{\theta} + d\right)\left(k_2 - k_1\right) + \underline{\theta} k_1 + c k_2^2}{2}
\end{aligned}
$$

and the corresponding profits are:

$$
\begin{aligned}
\Pi_1^* &= \frac{\left(\underline{\theta} k_1 - c k_1^2\right)\left(d\left(k_2 - k_1\right) + c k_2^2 - \underline{\theta} k_2\right)}{2d\left(k_2 - k_1\right)} \\
\Pi_2^* &= \frac{\left(c k_2^2 - d\left(k_2 - k_1\right) - \underline{\theta} k_2\right)^2}{4d\left(k_2 - k_1\right)}
\end{aligned}
$$

Proof. See Appendix. ■

In the corner solution, the price of the low-quality firm is equal to the gross surplus of the lowest valuation consumer, hence it does not depend on c and d. On the contrary, the price of the high-quality firm has the expected comparative statics, as it depends positively on c, d, $\underline{\theta}$ and $(k_2 - k_1)$.

4 Geometry of the price competition game

In this section, we also consider the cases where $k_2 < k_1 \leq \frac{1}{c}(\underline{\theta} + d)$.[9] To obtain the Nash equilibrium in these cases, we just need to invert the roles of firm 1 and firm 2. Thus the equilibrium and corresponding limits can be calculated by just changing the firms' indexes. We start by deriving an immediate consequence of the previous results.

[9] These cases are important to analyze any dynamic game where quality choices are also studied. Moreover this way we provide a complete characterization of the possible market configurations.

Corollary 1 *For given* $\underline{\theta}, c, d$ *and quality vector* (k_1, k_2) *with* $0 \leq k_i \leq \frac{1}{c}(\underline{\theta} + d)$ *for* $i = 1, 2$*, there exists a unique Nash equilibrium of the price game.*

Let us now explore the geometry of the different market configurations for different values of $\underline{\theta}$. Although the Nash equilibrium is unique, which type of Nash equilibrium holds for a given combination (k_1,k_2) depends on the values of $\underline{\theta}$, c and d. By representing graphically the conditions that have to be satisfied in each type of equilibrium, one can get a better perspective of when each market configuration holds. We did this for the most commonly used parameter values, $c = \frac{1}{2}$ and $d = 1$, for low, intermediate and high values of $\underline{\theta}$ (see Figure 4).

For $\underline{\theta} = 0$, the market is always partially covered as shown in Figure 4 (a). For $k_j \leq \frac{1}{c}(\underline{\theta} + d) - \frac{1}{2}k_i$, a partial coverage duopoly holds. For higher qualities, a low-quality constrained or unconstrained monopoly holds (above the diagonal firm 1 is the monopolist, below the diagonal firm 2 is the monopolist).

Figures 4 (b) and (c) show the market configurations for positive but low $\underline{\theta}$ ($\underline{\theta} \leq d = 1$). For these values of $\underline{\theta}$, the partial coverage configurations continue to be dominant. However as long as $\underline{\theta}$ is positive, for $k_i \leq \frac{\underline{\theta}}{c} = 2\underline{\theta}$ the Nash equilibrium is a duopoly with full coverage: an interior solution for relatively lower quality differentiation levels ((k_1, k_2) not very far from the diagonal) or a corner solution for higher differentiation levels. Looking at Figures 4 (b) and (c), we conclude that as $\underline{\theta}$ increases, the partial coverage duopoly region becomes smaller, whereas the reverse happens for the two duopoly full coverage configurations. Moreover, when the high-quality firm has a very high quality level, close to $\frac{\underline{\theta}+d}{c} = 2(\underline{\theta} + 1)$, the other firm behaves as a low-quality constrained or unconstrained monopolist, depending on how close the high quality is to $2(\underline{\theta} + 1)$ and the degree of differentiation (when differentiation is low, the monopolist is constrained by the surplus offered by the other firm).

Figures 4 (d) and (e) shows the market configurations for $d < \underline{\theta} \leq 2d$ (intermediate $\underline{\theta}$). For these values of $\underline{\theta}$, the three types of duopoly are still possible, although the partial coverage duopoly region becomes smaller, vanishing as $\underline{\theta}$ converges to $2d = 2$. Similarly, the regions of low-quality monopoly with partial coverage (constrained or unconstrained) continue to exist. However, for $\underline{\theta} > d = 1$, the high-quality monopoly becomes possible (it occurs when $k_1 + k_2 < \frac{\underline{\theta}-d}{c} = 2(\underline{\theta} - 1)$). Moreover, when the high-quality firm has a quality close to $\frac{\underline{\theta}+d}{c} = 2(\underline{\theta} + 1)$ and the low-quality firm has a quality below $\frac{\underline{\theta}-d}{c} = 2(\underline{\theta} - 1)$, the low-quality firm can behave as an unconstrained full coverage monopoly.

Figure 4 (f) shows the market configurations for high $\underline{\theta}$ (above $2d = 2$). For these values of $\underline{\theta}$, the partial coverage duopoly no longer exists. However, a new type of Nash equilibrium is possible: a low-quality full coverage constrained monopoly. This region is adjacent to the full coverage duopoly region, holding for quality combinations immediately above $\frac{1}{c}(\underline{\theta}+2d) = 2(\underline{\theta}+2)$.

Note that, as $\underline{\theta}$ increases, the duopoly regions cover a relatively smaller share of the possible market regions. The reason is that as $\underline{\theta}$ increases the relative dispersion among consumers becomes smaller and this implies that there are relatively fewer (k_1, k_2) that can lead to a duopoly solution.

The previous figures could be done for other values of c and d, by adjusting the scale. Note that the maximum quality that can be profitably offered is $\frac{\underline{\theta}+d}{c}$, the cutoff of the high-quality monopoly is $\frac{\underline{\theta}-d}{c}$, the interior full coverage region can never be above $(k_1, k_2) = (\frac{\underline{\theta}}{c}, \frac{\underline{\theta}}{c})$ and the frontier between the full coverage region and the low-quality full coverage constrained monopoly is $\frac{\underline{\theta}+2d}{c}$. Regarding these limits, changes in c can be interpreted as a rescaling of the figure and, for a given $\underline{\theta}$, this does not change the shape of the figure we get. However, in the case of the limits which are not linear, the impact of c on these limits is not a multiplicative one. Thus, changing c alters the relative size of the market regions that have non-linear limits. In other words, for $d = 1$ and a general c, to a great extent, we would have a simple rescaling of the previous figures, although the exact position of the curves changes with c.

The impact of d on the figures is different because it shifts the cutoff limits of $\underline{\theta}$. That is, the low $\underline{\theta}$, intermediate $\underline{\theta}$ and high $\underline{\theta}$ depend on d. For instance, if $d = 2$ the low $\underline{\theta}$ holds till $\underline{\theta} = 2$ and the figure we obtain for $\underline{\theta} = 2$ is similar to the one for $\underline{\theta} = 1$ and $d = 1$. Similarly, the figure for $\underline{\theta} = 4$ and $d = 2$ is similar to the one for $\underline{\theta} = 2$ and $d = 1$.

The next proposition summarizes the results regarding the possible market configurations:

Proposition 7 *The market equilibrium configurations depend on* $\underline{\theta}$ *as follows:*

1. *For* $\underline{\theta} = 0$*, either a partial coverage duopoly or a low-quality partial coverage monopoly (constrained or unconstrained) holds.*

2. *For* $0 < \underline{\theta} \leq d$*, either a duopoly (interior full coverage, corner full coverage or partial coverage) or a low-quality monopoly with partial coverage (constrained or unconstrained) holds.*

3. *For* $d < \underline{\theta} \leq 2d$*, either a duopoly (interior full coverage, corner full coverage or partial coverage), a low-quality partial coverage monopoly (constrained or unconstrained), a low-quality unconstrained full coverage monopoly or a high-quality constrained full coverage monopoly holds.*

4. *Finally, for* $\underline{\theta} > 2d$ *either a full coverage duopoly (interior or corner), a low-quality partial coverage monopoly (constrained or unconstrained), a low-quality full coverage (constrained or unconstrained) or a high-quality constrained full coverage monopoly holds.*

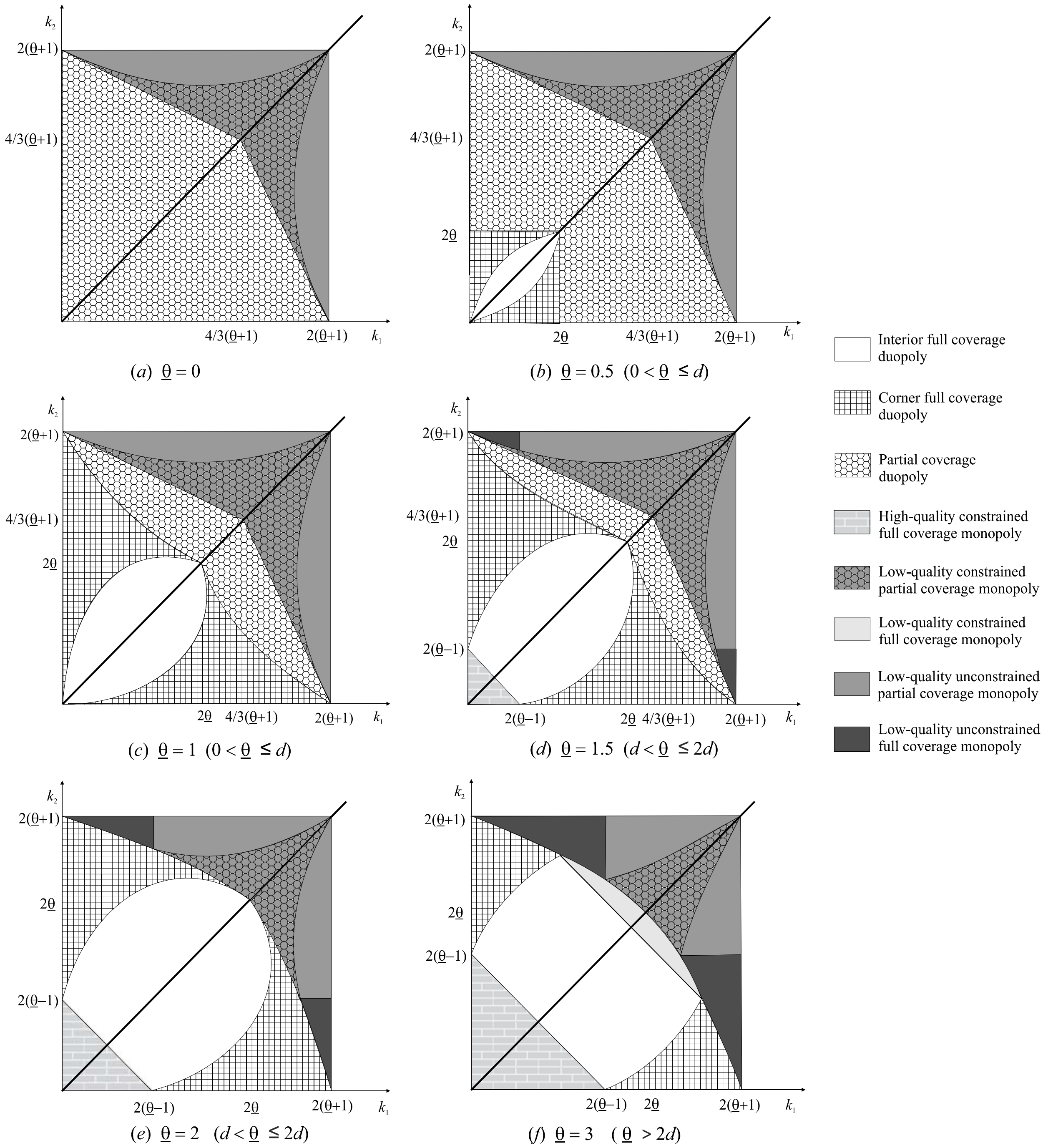

Figure 4: Market configurations for different values of $\underline{\theta}$, for $c = 0.5$ and $d = 1$. It should be noted that these figures do not have the same scale. All the figures consider qualities between zero and the maximum quality of each firm such that it has a positive demand, that is, $0 \leq k_i \leq \frac{1}{c}(\underline{\theta} + d) = 2(\underline{\theta} + 1)$. This way, each figure shows the relative size of each market region, instead of its absolute size.

Proof. See Appendix. ■

5 Conclusion

This paper provides a full characterization of the price game in a VPD model with quality dependent marginal production costs, for different values of the relevant parameters (lowest quality valuation, quality valuation dispersion and marginal cost coefficient). In a quite general setup, we show how the equilibrium market configuration depends on these parameters and on the quality combinations offered by the two firms.

Besides the classical cases of high-quality monopoly or duopoly with partial or full coverage, that have been addressed in the literature, we show the existence of a full coverage duopoly where the low-quality firm maximizes its profit in a corner solution, i.e., this firm offers a nil surplus to the lowest valuation consumer. This solution has been identified by Wauthy (1996) in the case of costless qualities, but we show it is also relevant when marginal production cost depends on quality. In addition, we show that both high-quality and low-quality monopolies are possible market configurations. The low-quality monopoly occurs when the high-quality firm has a very high quality. This gives a "surplus advantage" to the low-quality firm, as the high-quality firm has to incur a very high marginal cost, which implies that the surplus it can offer to consumers is lower than the one the low-quality firm can offer. If the "surplus advantage" is large, the low-quality firm may be able to behave as an unconstrained monopoly. However, for lower levels of "surplus advantage", to guarantee a monopoly, the low-quality firm has to charge a low enough price to match the surplus offered by the high-quality firm, thus the firm is a constrained monopolist. Similarly, when the lowest quality valuation is intermediate or high and the qualities offered by the two firms are low, the equilibrium market structure is a constrained full coverage high-quality monopoly. In this case, the high-quality firm is the one with "surplus advantage", because the low-quality firm offers a too low quality and hence offers a very small surplus that can be easily matched by the high-quality firm.

In most market configurations, the impact of the parameters' changes on the equilibrium prices and profits is the expected one, that is, prices are increasing with the marginal cost coefficient and with the consumers' quality valuation dispersion, and profits are decreasing with the marginal cost coefficient and increasing with the quality valuation dispersion. However, we show that these are not general results. There are some market configurations where prices or/and profits do not depend on one or both parameters. This happens, for instance, in the low-quality full coverage unconstrained monopoly, where the low-quality monopolist charges a price that gives nil surplus to the lowest valuation consumer. Thus, the price is not sensitive to the marginal cost coefficient and to the quality valuation dispersion, and the profit does not depend on the quality valuation dispersion. But, even more interestingly, there are cases where the results are the opposite of the expected: in the low-quality constrained full coverage monopoly, the profit is increasing with the marginal cost coefficient and decreasing with the quality valuation dispersion. This surprising result is due to the fact that the monopolist has to charge a price such that it matches the surplus offered by the competitor. Since the "surplus constraint" is binding, the impact of parameter changes on the low-quality monopoly price and profit is given through their impact on the "surplus constraint". When the marginal cost coefficient increases, there is an increase in the marginal cost of the competitor, which relaxes the constraint of the low-quality firm, allowing it to charge higher prices and to obtain higher profit. On the contrary, an increase in the quality valuation dispersion makes the constraint more stringent and the low-quality firm has to lower the price to match the surplus offered by the high-quality firm, hence the monopolist's profit decreases.

We determine the parameter combinations under which each price Nash equilibrium holds and show graphically the different market regions for different values of the valuation parameter, which reveals an amazing geometry. The market configuration that holds and how large it is depends on the lowest quality valuation and on the quality valuation dispersion. For instance, when the lowest quality valuation is «low», the predominant configurations are the ones with partial market coverage, and a high-quality monopoly cannot exist. On the contrary, for «high» lowest quality valuations, a duopoly with partial coverage is no longer possible, both high-quality and low-quality monopolies are possible and the monopoly configurations hold for a larger proportion of the possible quality combinations. The cutoff levels of «low», «intermediate» or «high» lowest quality valuation increase with the quality valuation dispersion. When quality valuation dispersion increases, a high-quality monopoly is only possible for higher levels of the lowest quality valuation parameter. Furthermore, partial coverage duopoly solutions are possible for higher levels of the lowest valuation consumer. Hence, increasing the quality valuation dispersion favours duopoly solutions and partial coverage solutions. This result is important both for firms' strategy, showing that, in order to increase the likelihood of monopoly solutions, they do not have an incentive to promote valuation dispersion (for instance through their marketing campaigns), as well as for antitrust authorities, who should be aware of these incentives. Policy makers who care about consumers' access to the product (in cases where this is socially relevant) should also consider that higher valuation dispersion does not promote full market coverage.

Our paper strengthens the importance of a complete analysis of the price stage game as a backbone for further analysis of dynamic games. Future work may consider dynamic VPD models, where firms first decide their quality (simultaneous or sequentially) and then make their pricing decisions.

Appendix – Proofs

Proof of Lemma 1. Assuming that $k_2 > k_1$, the difference in utilities for consumer θ, $U_1(\theta) - U_2(\theta)$, is positive if $(\theta k_1 - p_1) - (\theta k_2 - p_2) > 0$, or equivalently, $p_2 - p_1 > \theta (k_2 - k_1)$. Since the right hand side of the previous expression is increasing with θ, this implies that, if the condition holds for $(\underline{\theta} + d)$, then it holds for any $\theta < \underline{\theta} + d$. ■

Proof of Lemma 2. Assuming that $k_2 > k_1$, the difference in utilities for consumer θ, $U_2(\theta) - U_1(\theta)$, is positive if $(\theta k_2 - p_2) - (\theta k_1 - p_1) > 0$ or, equivalently, $p_2 - p_1 < \theta (k_2 - k_1)$. Since the right hand side of the previous expression is increasing with θ, this implies that, if the condition holds for $\underline{\theta}$, then it holds for any $\theta > \underline{\theta}$. ■

Proof of Lemma 3. Assuming that $k_2 > k_1$, from Lemma 1, we know that firm 2 can only have positive demand if the highest valuation consumer prefers the high-quality product. Moreover, from Lemma 2, we know that firm 1 can only have positive demand if the lowest valuation consumer prefers the low-quality product. This implies that $U_2(\theta) - U_1(\theta) = (\theta k_2 - p_2) - (\theta k_1 - p_1)$ must be negative at $\underline{\theta}$ but positive at $\underline{\theta} + d$. Since the function is continuous in θ, there exists an intermediate value of θ, $\widetilde{\theta}$, where $(\theta k_2 - p_2) - (\theta k_1 - p_1) = 0$. Moreover, since $U_2 - U_1$ is increasing in θ, then all consumers with $\theta > \widetilde{\theta}$ prefer to buy the product from firm 2, whereas all consumers with $\theta < \widetilde{\theta}$ prefer to buy it from firm 1. ■

Proof of Proposition 1. When $\widehat{\theta}_1(c_1) \leq \underline{\theta}$, or equivalently, $k_1 \leq \frac{\underline{\theta}}{c}$, to be a monopolist, firm 2 must cover the whole market and offer consumer $\underline{\theta}$ at least the same surplus he would get from firm 1. Hence, firm 2 solves the following problem:

$$\max_{p_2} \Pi_2 = (p_2 - c_2) \quad \text{subject to} \quad p_2 \leq \underline{\theta} (k_2 - k_1) + c_1$$

Since the profit function increases linearly with p_2, it is optimal to charge the highest possible price, thus $p_2^* = \underline{\theta} (k_2 - k_1) + c k_1^2$. For this to be the solution, the lowest valuation consumer has to get a non-negative surplus:

$$\underline{\theta} k_2 - \left(\underline{\theta} (k_2 - k_1) + c k_1^2 \right) \geq 0 \quad \Leftrightarrow \quad k_1 (\underline{\theta} - c k_1) \geq 0 \quad \Leftrightarrow \quad k_1 \leq \frac{\underline{\theta}}{c}.$$

For this to be the solution, firm 2 has to prefer to charge this low price and be a constrained monopolist than to charge a higher price and share the market with firm 1. Evaluating the derivative of firm 2's profit under a full coverage duopoly at $p_2 = \underline{\theta} (k_2 - k_1) + c_1$ and $p_1 = c_1$, we get (see the proof of Proposition 4):

$$\frac{\partial \Pi_2}{\partial p_2} = (d - \underline{\theta}) + c\,(k_1 + k_2)$$

The constrained monopoly is only optimal for firm 2 if this derivative is negative, i.e.:

$$k_1 + k_2 < \frac{\underline{\theta} - d}{c}$$

which can only hold for $\underline{\theta} > d$. Note that this condition is more restrictive than $k_1 \leq \frac{\underline{\theta}}{c}$. Thus, for this solution to hold, it is enough to assume $k_1 + k_2 < \frac{\underline{\theta}-d}{c}$. Considering the equilibrium price, $p_2^* = \underline{\theta}\,(k_2 - k_1) + ck_1^2$, the profit of firm 2 is given by:

$$\Pi_2^* = \underline{\theta}\,(k_2 - k_1) + ck_1^2 - ck_2^2 = (k_2 - k_1)\,(\underline{\theta} - c\,(k_1 + k_2))\,.$$

Lets us now consider the case where $\widehat{\theta}_1(c_1) > \underline{\theta}$ or, equivalently, $k_1 > \frac{\underline{\theta}}{c}$. In this case the "surplus constraint", is given by:

$$\widehat{\theta}_1(c_1)k_2 - p_2 \geq \widehat{\theta}_1(c_1)k_1 - c_1 \quad \Leftrightarrow \quad p_2 \leq c_1\frac{k_2}{k_1} = ck_1k_2$$

In this case, firm 2 may or may not cover the whole market and solves:

$$\max_{p_2} \Pi_2 = \frac{1}{d}\left(\underline{\theta} + d - \frac{p_2}{k_2}\right)(p_2 - c_2) \quad \text{subject to} \quad \underline{\theta}k_2 \leq p_2 \leq ck_1k_2$$

If none of the constraints is binding, the first-order condition for firm 2 is:

$$\frac{d\Pi_2}{dp_2} = \frac{1}{d}\left(-\frac{p_2 - c_2}{k_2} + \underline{\theta} + d - \frac{p_2}{k_2}\right) = 0 \quad \Leftrightarrow \quad p_2^* = \frac{(\underline{\theta} + d)\,k_2 + ck_2^2}{2},$$

and the corresponding equilibrium profits are:

$$\Pi_1^* = 0 \quad \text{and} \quad \Pi_2^* = \frac{k_2\,(\underline{\theta} + d - ck_2)^2}{4d}.$$

For this to be the solution, the lowest valuation consumer has to get a negative surplus, if buying from firm 2, as otherwise the market would be fully covered:

$$U_2(\underline{\theta}) < 0 \;\Leftrightarrow\; \underline{\theta}k_2 - \left(\frac{(\underline{\theta} + d)\,k_2 + ck_2^2}{2}\right) < 0 \;\Leftrightarrow\; k_2 > \frac{\underline{\theta} - d}{c}$$

which is satisfied as $k_2 > k_1 \geq \frac{\underline{\theta}}{c}$. Moreover, the "surplus constraint" cannot be binding:

$$\frac{(\underline{\theta}+d)\,k_2 + ck_2^2}{2} < ck_1k_2 \quad \Leftrightarrow \quad k_2\left((\underline{\theta}+d) + ck_2 - 2ck_1\right) < 0 \quad \Leftrightarrow \quad k_2 < 2k_1 - \frac{\underline{\theta}+d}{c}$$

This implies that, if $k_1 > \frac{\underline{\theta}}{c}$ and $k_1 \leq k_2 < 2k_1 - \frac{\underline{\theta}+d}{c}$, none of the constraints is binding and the high-quality firm can behave as an unconstrained monopolist. However, for this to be possible it must be $k_1 < 2k_1 - \frac{\underline{\theta}+d}{c}$ or, equivalently, $k_1 > \frac{\underline{\theta}+d}{c}$, which is not possible. If $k_2 \geq 2k_1 - \frac{\underline{\theta}+d}{c}$ the previous solution is no longer valid as the "surplus constraint" would be binding. In that case, $p_2^* = ck_1k_2$ and the profits of firm 2 are

$$\Pi_2^* = \frac{1}{d}\left(\underline{\theta}+d-ck_1\right)\left(ck_1k_2 - ck_2^2\right) = \frac{\left(\underline{\theta}+d-ck_1\right)(k_1-k_2)ck_2}{d}$$

which can only be positive for $k_1 > \frac{\underline{\theta}+d}{c}$, which again is not possible as we are assuming $k_1 < k_2 < \frac{\underline{\theta}+d}{c}$. So this solution cannot hold. Therefore, we cannot have a high-quality firm monopoly with partial coverage. ■

Proof of Proposition 2. If $c_2 - (\underline{\theta}+d)\,(k_2-k_1) \leq \underline{\theta}k_1$, firm 1 solves the following problem:

$$\max_{p_1} \Pi_1 = (p_1 - c_1) \qquad \text{subject to} \quad p_1 \leq c_2 - (\underline{\theta}+d)\,(k_2-k_1)$$

Since Π_1 increases with p_1, it is optimal for firm 1 to set the highest price possible, $p_1^* = c_2 - (\underline{\theta}+d)\,(k_2-k_1)$. Thus the "surplus constraint" is binding. Therefore, for firm 1 to behave as an unconstrained monopolist, we must have $c_2 - (\underline{\theta}+d)\,(k_2-k_1) > \underline{\theta}k_1$ or, equivalently, $k_2 < \frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c}$ or $k_2 > \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-4cdk_1}}{2c}$. Assuming that these conditions hold, firm 1 solves the following problem:

$$\max_{p_1} \Pi_1 = \frac{1}{d}\left(\underline{\theta}+d-\frac{p_1}{k_1}\right)(p_1-c_1) \qquad \text{subject to} \quad \underline{\theta}k_1 \leq p_1 \leq c_2 - (\underline{\theta}+d)\,(k_2-k_1)$$

If none of the constraints is binding in this optimization problem, the first-order condition is:

$$\frac{d\Pi_1}{dp_1} = \frac{1}{d}\left(-\frac{p_1-c_1}{k_1} + \underline{\theta} + d - \frac{p_1}{k_1}\right) = 0$$

Solving it with respect to p_1, we obtain:

$$p_1^* = \frac{ck_1^2 + k_1(\underline{\theta}+d)}{2}$$

For this to be the solution of firm 1's problem and partial coverage to occur:

$$
\begin{aligned}
p_1^* &> \underline{\theta} k_1 \quad \Leftrightarrow \quad k_1 \left(ck_1 + d - \underline{\theta}\right) > 0 \quad \Leftrightarrow \quad k_1 > \frac{\underline{\theta} - d}{c} \\
p_1^* &< ck_2^2 - \left(\underline{\theta} + d\right)\left(k_2 - k_1\right) \quad \Leftrightarrow \quad ck_1^2 - 2ck_2^2 + \left(\underline{\theta} + d\right)\left(2k_2 - k_1\right) < 0
\end{aligned}
$$

Solving the last condition with respect to k_2, we conclude that the condition is satisfied if

$$
k_2 < \frac{(\underline{\theta} + d) - \sqrt{(\underline{\theta} + d)^2 - 2c\left((\underline{\theta} + d)k_1 - ck_1^2\right)}}{2c} \text{ or } k_2 > \frac{(\underline{\theta} + d) + \sqrt{(\underline{\theta} + d)^2 - 2c\left((\underline{\theta} + d)k_1 - ck_1^2\right)}}{2c}
$$

It is easy to show that $(\underline{\theta} + d)^2 - 2c\left((\underline{\theta} + d)k_1 - ck_1^2\right) \geq (\underline{\theta} + d)^2 - 4cdk_1$ if and only if $k_1 \geq \frac{\underline{\theta} - d}{c}$. Therefore, if $k_1 > \frac{\underline{\theta} - d}{c}$ and $k_2 > \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c}$ or if $k_2 < \frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c}$ firm 1 behaves as an unconstrained monopolist and covers partially the market.

If $k_1 \leq \frac{\underline{\theta} - d}{c}$, with the previous price the lowest valuation consumer would get a non-negative surplus, but then the constraint $\underline{\theta} k_1 \leq p_1$ is binding, the solution to firm 1's optimization problem is $p_1^* = \underline{\theta} k_1$ and the market is fully covered. Note that, in this case, the "surplus constraint" cannot be binding and, again, firm 1 is behaving as an unconstrained monopolist.■

Proof of Proposition 3. When $c_2 - (\underline{\theta} + d)(k_2 - k_1) \leq \underline{\theta} k_1$, the lowest valuation consumer gets a non-negative surplus if buying from firm 1, if firm 1 charges $p_1 \leq c_2 - (\underline{\theta} + d)(k_2 - k_1)$. To guarantee a monopoly, firm 1 solves the following problem:

$$
\max_{p_1} \Pi_1 = (p_1 - c_1) \quad \text{subject to} \quad p_1 \leq c_2 - (\underline{\theta} + d)(k_2 - k_1)
$$

Since Π_1 increases with p_1, it is optimal for firm 1 to set the highest price possible, $p_1^* = c_2 - (\underline{\theta} + d)(k_2 - k_1)$. Thus, the "surplus constraint" is binding. Thus, when $\frac{1}{c}\left((\underline{\theta} + d) - \sqrt{(\underline{\theta} + d)^2 - 4cdk_1}\right) \leq k_2 \leq \frac{1}{c}\left((\underline{\theta} + d) + \sqrt{(\underline{\theta} + d)^2 - 4cdk_1}\right)$, the market is fully covered and the "surplus constraint" is binding. Since at this price firm 2 is getting zero demand, the profit of firm 1 is precisely the same under monopoly and under duopoly. Let us see if firm 1 would gain by increasing slightly the price, which would imply becoming a duopolist, by analyzing the derivative of firm 1's profit function. Under full coverage, the derivative of the firm 1's profit function (see proof of Proposition 4) at prices $p_1^* = c_2 - (\underline{\theta} + d)(k_2 - k_1)$ and $p_2^* = c_2$ is:

$$
\frac{\partial \Pi_1}{\partial p_1}(p_1^*, p_2^*) = -\underline{\theta} + 2d - c(k_1 + k_2)
$$

Thus, under full coverage, firm 1 prefers to be a constrained monopoly only if this derivative is negative, i.e.:

$$k_1 + k_2 > \frac{\underline{\theta} + 2d}{c}.$$

In the partial coverage constrained monopoly case we know that the "surplus constraint" is binding for $\frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c} \leq k_2 \leq \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c}$. Evaluating the derivative of firm 1 under partial coverage duopoly (see proof of Proposition 5) at prices $p_1^* = c_2 - (\underline{\theta} + d)(k_2 - k_1)$ and $p_2^* = c_2$ we get:

$$\frac{\partial \Pi_1}{\partial p_1}(p_1^*, p_2^*) = \frac{k_2 \left(2(\underline{\theta} + d) - c(k_1 + 2k_2)\right)}{dk_1}$$

If this derivative is negative, firm 1 prefers to be a monopolist with constrained price than increase the price and share the market with firm 2. The derivative is negative for:

$$c(k_1 + 2k_2) > 2(\underline{\theta} + d).$$

Thus for $\frac{(\underline{\theta}+d)-\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c} \leq k_2 \leq \frac{(\underline{\theta}+d)+\sqrt{(\underline{\theta}+d)^2-2c\left((\underline{\theta}+d)k_1-ck_1^2\right)}}{2c}$ and $k_1 + 2k_2 > \frac{2}{c}(\underline{\theta}+d)$, the Nash equilibrium of the price game is constrained monopoly with partial coverage.■

Proof of Proposition 4. The first-order conditions of the two firms profit maximization problems are:

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial p_1} &= \frac{1}{d}\left(-\frac{p_1 - c_1}{k_2 - k_1} + \left(\frac{p_2 - p_1}{k_2 - k_1} - \underline{\theta}\right)\right) = 0 \\
\frac{\partial \Pi_2}{\partial p_2} &= \frac{1}{d}\left(-\frac{p_2 - c_2}{k_2 - k_1} + \left(\underline{\theta} + d - \frac{p_2 - p_1}{k_2 - k_1}\right)\right) = 0
\end{aligned}$$

Note that $\frac{\partial^2 \Pi_i}{\partial p_i^2} = \frac{-2}{d(k_2 - k_1)} < 0$, thus the second-order conditions of the two firms optimization problems are both satisfied. Solving the previous system with respect to p_1 and p_2 and replacing c_i by ck_i^2, we obtain the interior solution Nash equilibrium prices:

$$p_1^* = \frac{(d - \underline{\theta})(k_2 - k_1) + 2ck_1^2 + ck_2^2}{3} \quad \text{and} \quad p_2^* = \frac{(\underline{\theta} + 2d)(k_2 - k_1) + 2ck_2^2 + ck_1^2}{3}.$$

Substituting the equilibrium prices into the profit functions and simplifying them, we get the equilibrium profits. Note that this interior solution is only valid if, considering the equilibrium prices, the lowest valuation consumer obtains a non-negative surplus, as otherwise the market would not be fully covered and the profit function of firm 1 would be different. Hence, the interior solution is valid if and only if:

$$U(\underline{\theta}) = \underline{\theta} k_1 - p_1^* \geq 0 \quad \Leftrightarrow \quad \frac{k_1}{3}\left(2\underline{\theta} + d - 2ck_1\right) + \frac{k_2}{3}\left(\underline{\theta} - d - ck_2\right) \geq 0$$

In equilibrium, the indifferent consumer between buying from firm 1 or from firm 2 equals:

$$\widetilde{\theta}(p_1^*, p_2^*) = \frac{p_2^* - p_1^*}{k_2 - k_1} = \frac{2\underline{\theta} + d + c(k_1 + k_2)}{3}$$

That is, $\widetilde{\theta}$ depends on the lowest valuation $\underline{\theta}$, the dispersion coefficient and the sum of the qualities of the two firms. For this to be the equilibrium, we must have $\underline{\theta} \leq \widetilde{\theta}(p_1^*, p_2^*) \leq \underline{\theta} + d$, which is is equivalent to:

$$\frac{\underline{\theta} - d}{c} \leq k_1 + k_2 \leq \frac{\underline{\theta} + 2d}{c}.\blacksquare$$

Proof of Proposition 5 In an interior solution, the first-order conditions of the two firms profit maximization problems are:

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial p_1} &= \frac{1}{d}\left(-\frac{p_1 - c_1}{k_2 - k_1} - \frac{p_1 - c_1}{k_1} + \frac{p_2 - p_1}{k_2 - k_1} - \frac{p_1}{k_1}\right) = 0 \\
\frac{\partial \Pi_2}{\partial p_2} &= \frac{1}{d}\left(-\frac{p_2 - c_2}{k_2 - k_1} + \underline{\theta} + d - \frac{p_2 - p_1}{k_2 - k_1}\right) = 0.
\end{aligned}$$

Note that $\frac{\partial^2 \Pi_1}{\partial p_1^2} = -\frac{2}{d(k_2 - k_1)} - \frac{2}{dk_1} < 0$ and $\frac{\partial^2 \Pi_2}{\partial p_2^2} = -\frac{2}{d(k_2 - k_1)} < 0$, thus the second-order conditions are satisfied. Solving the system of equations with respect to p_1 and p_2, we obtain:

$$p_1^* = \frac{k_1(\underline{\theta} + d)\left(k_2 - k_1\right) + 2c_1 k_2 + c_2 k_1}{4k_2 - k_1} \quad \text{and} \quad p_2^* = \frac{2k_2(\underline{\theta} + d)\left(k_2 - k_1\right) + c_1 k_2 + 2c_2 k_2}{4k_2 - k_1}$$

Substituting the marginal costs, $c_i = ck_i^2$, we get the equilibrium prices as a function of the qualities:

$$p_1^* = \frac{k_1(\underline{\theta} + d)\left(k_2 - k_1\right) + 2ck_1^2 k_2 + ck_2^2 k_1}{4k_2 - k_1} \quad \text{and} \quad p_2^* = \frac{2k_2(\underline{\theta} + d)\left(k_2 - k_1\right) + ck_1^2 k_2 + 2ck_2^3}{4k_2 - k_1}.$$

Substituting in the profit functions, we obtain the equilibrium profits:

$$\Pi_1^* = \frac{k_1 k_2 (k_2 - k_1) (\underline{\theta} + d + c (k_2 - k_1))^2}{d (4k_2 - k_1)^2}$$
$$\Pi_2^* = \frac{k_2^2 (k_2 - k_1) (c (2k_2 + k_1) - 2 (\underline{\theta} + d))^2}{d (4k_2 - k_1)^2}$$

For this partial coverage solution to hold, the condition $p_1 > \underline{\theta} k_1$ has to be satisfied. This is equivalent to:

$$\underline{\theta} k_1 - \frac{k_1 (\underline{\theta} + d) (k_2 - k_1) + 2ck_1^2 k_2 + ck_2^2 k_1}{4k_2 - k_1} < 0 \Leftrightarrow \frac{k_1}{4k_2 - k_1} (3\underline{\theta} k_2 - d (k_2 - k_1) - ck_2 (2k_1 + k_2)) < 0$$

As $\frac{k_1}{4k_2 - k_1} > 0$, this is equivalent to:

$$d (k_2 - k_1) + ck_2 (2k_1 + k_2) - 3\underline{\theta} k_2 > 0$$

In addition, for this to be a Nash equilibrium, it must be that, considering the equilibrium price p_1^*, the consumer which is indifferent between buying from firm 1 and not buying, $\hat{\theta}_1(p_1^*) \leq \widetilde{\theta}(p_1^*, p_2^*)$, and also that $\widetilde{\theta}(p_1^*, p_2^*) \leq (\underline{\theta} + d)$, since otherwise all the consumers would prefer firm 1. In equilibrium, $\hat{\theta}_1(p_1^*)$ and $\widetilde{\theta}(p_1^*, p_2^*)$ are given by:

$$\widetilde{\theta}(p_1^*, p_2^*) = \frac{2 (\underline{\theta} + d) k_2 - (\underline{\theta} + d) k_1 + 2ck_2^2 + ck_1 k_2}{4k_2 - k_1}$$
$$\hat{\theta}_1(p_1^*) = \frac{(\underline{\theta} + d) k_2 - (\underline{\theta} + d) k_1 + ck_2^2 + 2ck_1 k_2}{4k_2 - k_1}$$

Thus, for this solution to hold

$$\widetilde{\theta}(p_1^*, p_2^*) - \hat{\theta}_1(p_1^*) \geq 0 \quad \Leftrightarrow \quad k_2 ((\underline{\theta} + d) + c (k_2 - k_1)) \geq 0$$

which is always true as we are assuming $k_2 > k_1$. Finally:

$$\widetilde{\theta}(p_1^*, p_2^*) - (\underline{\theta} + d) \leq 0 \quad \Leftrightarrow \quad \frac{k_2 (c(k_1 + 2k_2) - 2 (\underline{\theta} + d))}{4k_2 - k_1} \leq 0 \quad \Leftrightarrow \quad k_2 \leq \frac{1}{c} (\underline{\theta} + d) - \frac{1}{2} k_1. \blacksquare$$

Proof of Proposition 6. This case can only hold if neither the full coverage nor the partial coverage duopoly hold. Therefore, $\frac{k_1}{3} (2\underline{\theta} + 1 - k_1) + \frac{k_2}{6} (2\underline{\theta} - 2 - k_2) < 0$ and $d (k_2 - k_1) - 3\underline{\theta} k_2 + ck_2 (2k_1 + k_2) \leq 0$. The first-order condition of firm 2 is the same as in the two previous cases:

$$\frac{\partial \Pi_2}{\partial p_2} = \frac{1}{d}\left(-\frac{p_2 - c_2}{k_2 - k_1} + \underline{\theta} + d - \frac{p_2 - p_1}{k_2 - k_1}\right) = 0$$

On the other hand, firm 1's constraint is binding and thus the best response of firm 1 is $p_1 = \underline{\theta} k_1$. Solving the system of best responses and substituting the marginal costs $c_i = ck_i^2$, we get the equilibrium prices:

$$p_1^* = \underline{\theta} k_1 \quad \text{and} \quad p_2^* = \frac{(\underline{\theta} + d)(k_2 - k_1) + \underline{\theta} k_1 + ck_2^2}{2}$$

Substituting in the profit functions and simplifying them, we obtain

$$\Pi_1^* = \frac{\left(\underline{\theta} k_1 - ck_1^2\right)\left(d(k_2 - k_1) + ck_2^2 - \underline{\theta} k_2\right)}{2d(k_2 - k_1)}$$

$$\Pi_2^* = \frac{\left(ck_2^2 - d(k_2 - k_1) - \underline{\theta} k_2\right)^2}{4d(k_2 - k_1)}$$

In this solution, we already assumed that the lowest valuation consumer has nil utility. However, in order for this to be the equilibrium, we still have to check if $\underline{\theta} \leq \widetilde{\theta}(p_1^*, p_2^*) \leq \underline{\theta} + d$ (so that both firms operate). The indifferent consumer is given by:

$$\widetilde{\theta}(p_1^*, p_2^*) = \frac{d}{2} + \frac{\underline{\theta}(k_2 - 2k_1) + ck_2^2}{2(k_2 - k_1)}$$

Thus $\underline{\theta} \leq \widetilde{\theta}(p_1^*, p_2^*) \leq \underline{\theta} + d$ is equivalent to:

$$\underline{\theta} - \frac{d}{2} \leq \frac{\underline{\theta}(k_2 - 2k_1) + ck_2^2}{2(k_2 - k_1)} \leq \underline{\theta} + \frac{d}{2}. \blacksquare$$

Proof of Proposition 7. The results are a consequence of the conditions that have to be satisfied for each type of equilibrium to hold. In the interior full coverage duopoly, the conditions that guarantee that the market is fully covered, considering both the case of $k_1 < k_2$ and $k_1 > k_2$ are

$$\frac{k_1}{3}(2\underline{\theta} + d - 2ck_1) + \frac{k_2}{3}(\underline{\theta} - d - ck_2) \geq 0$$

$$\frac{k_2}{3}(2\underline{\theta} + d - 2ck_1) + \frac{k_1}{3}(\underline{\theta} - d - ck_2) \geq 0$$

These two curves intersect at $(k_1, k_2) = (0, 0)$ and $(k_1, k_2) = (\frac{\underline{\theta}}{c}, \frac{\underline{\theta}}{c})$, and the full coverage duopoly holds between the two curves. However, we also need to consider the conditions that guarantee that the indifferent consumer is between $\underline{\theta}$ and $\underline{\theta}+d$, as they may be binding:

$$\frac{1}{c}\left(\underline{\theta}-d\right) \leq k_2+k_1 \leq \frac{1}{c}\left(\underline{\theta}+2d\right)$$

For $\underline{\theta} < d$ the lower bound is negative, and is therefore not relevant as qualities have to be non-negative. However, if $\underline{\theta} > d$, $k_1+k_2 \geq \frac{1}{c}\left(\underline{\theta}-d\right) > 0$, the lower bound condition becomes relevant to identify the interior full coverage duopoly. Note that, according to Proposition 1, if $\underline{\theta} > d$ and $k_1+k_2 < \frac{1}{c}\left(\underline{\theta}-d\right)$, a constrained full coverage high-quality holds. This shows the first important cutoff in the Nash equilibrium market regions, $\underline{\theta}=d$, as the market regions that hold depend on being below or above $\underline{\theta}$ (and the conditions that are binding to define the interior full coverage duopoly also depend on that).

Let us now look at the upper bound, $k_2+k_1 \leq \frac{1}{c}\left(\underline{\theta}+2d\right)$. For this constraint to be relevant, it has to be below $(k_1,k_2)=(\frac{\underline{\theta}}{c},\frac{\underline{\theta}}{c})$, which implies $k_2+k_1=\frac{2\underline{\theta}}{c}$, which can only happen if

$$\frac{\underline{\theta}+2d}{c} \leq \frac{2\underline{\theta}}{c} \quad \Leftrightarrow \quad \underline{\theta} \geq 2d$$

So, for $\underline{\theta} \geq 2d$, the upper bound becomes relevant in identifying the interior full coverage region. Note that, according to Proposition 3, if $k_2+k_1 > \frac{1}{c}\left(\underline{\theta}+2d\right)$ a constrained full coverage low-quality monopoly starts holding. This is also the cutoff above which we no longer have a partial coverage duopoly.

We also confirmed the results through numerical simulations with extremely small steps in k_1 and k_2, for each value of $\underline{\theta}$, which allowed us to get the matrices of market configurations for each $\underline{\theta}$. These matrices are completely coherent with the figures obtained from the analytical solution.■

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