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Modelling time-varying volatility interactions

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ABSTRACT

We propose an additive time-varying (or partially time-varying) multivariate model of volatility, where a time-dependent component is added to the extended vector GARCH process for modelling the dynamics of volatility interactions. Volatility co-dependence is allowed to change smoothly between two extreme states, and second-moment interdependence is identified through these structural changes. The estimation of the new time-varying vector GARCH process is simplified using an equation-by-equation estimator for the volatility equations in the first step and estimating the correlation matrix in the second step. A new Lagrange multiplier test is derived for testing the null hypothesis of constant volatility co-dependence against a smoothly time-varying interdependence between financial markets. Monte Carlo experiments show that the test statistic has satisfactory finite-sample properties. An empirical application to sovereign bond yields illustrates the modelling strategy and the usefulness of the new specification.

1. Introduction

The class of conditional correlation (CC-)GARCH models of financial time series has become a standard tool for modelling and forecasting correlations among financial returns. Following the constant conditional correlation (CCC-)GARCH model of Bollerslev (1990) many parametric extensions have been proposed in the literature on building more flexible models to describe time-varying conditional correlations. The models introduced by Tse and Tsui (2002) and Engle (2002) with dynamic conditional correlations postulating GARCH-type dynamics on the correlations have become particularly popular among practitioners. For surveys on these and other multivariate GARCH models, see Bauwens et al. (2006) and Silvennoinen and Teräsvirta (2009).

As a result of financial market connectivity, the analysis of interdependence and interactions in volatility is useful to gain knowledge about how information is transmitted across markets. Understanding the mechanism of transmission of financial market movements is important for portfolio risk management and for successful hedging and trading strategies. Several studies have documented evidence of interdependence and links between financial markets or assets. Empirical studies that provide evidence for volatility spillovers include Baillie and Bollerslev (1990), King and Wadhwani (1990), Hamao et al. (1990), Lin et al. (1994), Cifarelli and Paladino (2005), among others.

Despite the extensive literature investigating co-movements among financial markets, most research has focused on interdependence in the conditional first moments of the distribution of returns, and less attention has been devoted to exploring financial interactions in terms of the second moments. Some examples of the former include (Diebold & Yilmaz, 2009) who used vector autoregressive methods to examine the transmission of financial market movements. Chiang and Wang (2011) proposed an autoregressive range-based volatility model and used a smooth transition copula function to further examine financial volatility contagion between financial stock markets of the G7 countries. Leung et al. (2017) used a linear regression approach to study volatility spillovers (or interactions) between the GARCH volatilities of the equity and exchange rate markets during periods of financial crises. However, the findings in Engle and Susmel (1993) and Diebold and Yilmaz (2009) suggest that cross-market volatility linkages provide more insightful information about the dynamics of co-movements across markets than linkages in returns.

Several specification techniques have been used in the literature to examine the dynamics of financial market interdependence. Only a few attempts have been made to investigate the time-varying nature of volatility spillovers. Building on conditional correlations, Jeantheau (1998) generalised the diagonal CCC-GARCH model to the so-called extended (E)CCC-GARCH model where dynamic volatility interactions between markets are allowed in the form of cross-market ARCH and GARCH effects. However, recent research found evidence for time-variation and structural shifts in the transmission mechanisms of shocks

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to volatility during periods of financial market distress. Notable works include those by Caporin and Malik (2020) and Malik (2022), who emphasised the consequences of ignoring nonstationary volatility by showing that it can lead to spurious volatility transmission as structural changes in the economy, fundamentals, or crises can alter the dynamics among markets. Other examples documenting time-varying volatility spillovers can be found in Karanasos et al. (2014), Jung and Maderitsch (2014), Karanasos et al. (2018) and Liu and Gong (2020), among others. The contribution of this work is to propose a new model specification in this literature that allows for time-varying volatility spillovers.

Although conditional correlation GARCH models have been widely used to quantify and examine volatility spillovers, less attention has been paid to potential misspecifications in their GARCH structure. The contribution of this paper lies in that direction. Testing the assumption of constant volatility spillovers between time series is an important specification tool for building multivariate GARCH models. Modelling volatility spillovers would be relevant only when the null hypothesis of no interactions is rejected. For a comprehensive discussion of tests for volatility spillovers, see, for example, Hong (2001), Nakatani and Teräsvirta (2009) and Pedersen (2017). Other specification issues of interest include alleviating the curse of dimensionality in large systems or developing models whose parametric structure is well suited for time-varying co-movements across markets.

In the vast existing literature on multivariate volatility models, there are still some unanswered questions and issues that remain unsolved. Some of these challenges are addressed in this work as follows. Firstly, the majority of vector GARCH processes have been designed under the assumption of weak and strict stationarity. However, this condition may not be suitable when modelling long economic processes, as structural changes are likely to occur over long periods. Therefore, our modelling procedure extends the vector GARCH process by allowing a more flexible specification that accounts for nonstationarity. Secondly, it is well known that the conditional variance changes with the arrival of market-specific news or volatility spillovers across markets; see e.g., Engle et al. (1990), Tchahou and Duchesne (2013), and Hafner and Herwartz (2006). Practitioners are particularly interested in understanding changes in volatility, as these are closely linked to the arrival of news and the resulting market dynamics. Our formulation allows for a parametric specification to identify the presence of causality in variance and to test how changes in the volatility of a given asset or market affect the volatility of other assets or markets, under nonstationarity. Structural instability in volatility spillovers between calm periods and periods of market distress has been documented in Karanasos et al. (2014) Karanasos et al. (2018) and Liu and Gong (2020), among others. One advantage of new parametric specification over existing ones is its flexibility in modelling the evolving behaviour of shock-transmission mechanisms, depending on the state of the economy. Finally, a statistical testing procedure is designed to detect time-varying causality in variance and identify the direction of causation in the presence of nonstationarity.

This paper contains two novelties. First, we propose a novel extension of the ECCC-GARCH representation of Jeantheau (1998) suited to model time-varying conditional variances and cross-market volatility interactions. It is based on an additive decomposition of the conditional variance equations into two components, one describing clustering volatility specified as a measurable function of the past of all elements of the vector of returns and another representing the misspecification of the volatility structure. One type of model misspecification includes omitting a time-dependent component to the extended vector GARCH model in the form of structural changes in the volatility process. Other types of model misspecification in the functional form of conditional variances could also be considered. Second, we introduce statistical tests based on the Lagrange multiplier principle as a useful validation tool to reveal against such type of model misspecification. Monte Carlo simulations show that the tests exhibit reasonable good size and power in finite samples.

Our modelling strategy relies first on testing the adequacy of the specification against the alternative of the additive time-dependent vector GARCH model and estimating such nonlinear extension only in case of rejection of the null hypothesis. For numerical simplicity, we shall adopt the two-step estimation approach proposed by Francq and Zakoïan (2016) where the univariate conditional variances are estimated equation-by-equation in the first step and the conditional correlations are estimated in the second step conditionally on the first step estimates. Finally, we shall illustrate our modelling cycle with an application to study the dynamics of the co-movements among bond yield returns on the 10-year government Greek, Irish, and Portuguese sovereign markets.

The remainder of the paper is organised as follows. Section 2 introduces the new model and its properties. In Section 3, the equation-by-equation estimation of parameters is presented and the asymptotic properties of the quasi-maximum likelihood estimator (QMLE) are discussed. Section 4 is devoted to Lagrange multiplier tests for time-varying volatility interactions. Section 5 presents the modelling strategy of the new specification. Section 6 presents some evidence of the small-sample properties of the statistical tests using Monte Carlo simulations. In Section 7, we illustrate the functioning of the modelling strategy empirically using sovereign bond yields. Finally, Section 8 concludes the paper.

2. The model and assumptions

Consider the observable stochastic m−dimensional vector of returns

$$\epsilon_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\zeta}_t = \mathbf{D}_t\mathbf{z}_t, \qquad t = 1,\dots,T, \tag{1}$$

where the stochastic vector $\mathbf{z}_t = (z_{1t},\dots,z_{mt})'$ is a sequence of independent random variables with $\mathsf{E}\mathbf{z}_t = \mathbf{0}$ and a positive definite, potentially time-varying covariance matrix $\mathsf{E}\mathbf{z}_t\mathbf{z}_t' = \mathbf{P}_t = [\rho_{ij,t}]$, such that $\rho_{ii,t} = 1, \forall i$, and $\rho_{ij,t} \neq 0, \forall i \neq j,\ i,j = 1,\dots,m$. Many different forms are possible for $\mathbf{P}_t$, but in order to facilitate the exposition we shall focus on the constant correlation structure, that is, $\mathbf{P}_t = \mathbf{P} = [\rho_{ij}]$. It follows that the error vector $\boldsymbol{\zeta}_t = \mathbf{P}^{-1/2}\mathbf{z}_t \sim \text{iid}(\mathbf{0},\mathbf{I}_m)$, where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix. For simplicity, we assume the conditional mean of the vector of returns is equal to zero. The conditional covariance matrix $\boldsymbol{\Sigma}_t = [\sigma_{ij,t}]$ of ϵ_t is assumed to have the multiplicative decomposition:

$$\boldsymbol{\Sigma}_t = \mathbf{D}_t\mathbf{P}\mathbf{D}_t \tag{2}$$

where $\mathbf{D}_t = \mathsf{diag}(\sigma_1(t/T),\dots,\sigma_m(t/T))$ is a diagonal matrix of conditional standard deviations of the process ϵ_t, T is the sample size and $\mathbf{P} = [\rho_{ij}]$ is the conditional correlation matrix for ϵ_t whose elements ρ_{ij} are constant for $\forall i \neq j$. Each component $\sigma_i(t/T)$, $i = 1,\dots,m$, is a smooth time-dependent function, describing structural changes in the conditional variance. With these assumptions, the vector process ϵ_t is a martingale-difference

$$\mathsf{E}(\epsilon_t|\mathcal{F}_{t-1}) = \mathbf{0} \tag{3}$$

with a symmetric conditional covariance matrix defined as

$$\mathsf{E}(\epsilon_t\epsilon_t'|\mathcal{F}_{t-1}) = \boldsymbol{\Sigma}_t \tag{4}$$

where $\mathcal{F}_{t-1}$ is the σ−algebra generated by past information about ϵ_t available at time $t-1$.

In this paper, we generalise the class of conditional correlation GARCH models by introducing nonstationarity in the own-market volatility process and cross-market volatility interactions. We rely on statistical inference to specify the most appropriate parameterisation of $\sigma_i^2(t/T)$ so that it fully captures past information. Further details about the statistical test are discussed in Section 4. We assume that $\boldsymbol{\sigma}_t^2 = (\sigma_{1t}^2,\dots,\sigma_{mt}^2)'$ is defined as a time-varying representation, measurable with respect to $\mathcal{F}_{t-1}$, with additive decomposition:

$$\boldsymbol{\sigma}_t^2 = \mathbf{h}_t + \mathbf{g}_t \tag{5}$$

where $\mathbf{h}_t$ is the stationary component that allows for volatility interactions among markets, and $\mathbf{g}_t$ is the time-dependent volatility component of rescaled time. This rescaling technique for calendar time is useful for obtaining a meaningful asymptotic theory (Dahlhaus & Rao, 2006), but establishing the asymptotic properties of the maximum likelihood estimators is beyond the scope of this paper.

Let $\mathbf{A}_r$ and $\mathbf{B}_s$ be $m \times m$ matrices with positive entries, where $\mathbf{A}_r = (\boldsymbol{\alpha}'_{1,r}, \dots, \boldsymbol{\alpha}'_{m,r})'$, with $\boldsymbol{\alpha}_{i,r} = (\alpha_{i1,r}, \dots, \alpha_{im,r})$ is the ith row of $\mathbf{A}_r$, and $\mathbf{B}_s = (\boldsymbol{\beta}'_{1,s}, \dots, \boldsymbol{\beta}'_{m,s})'$, with $\boldsymbol{\beta}_{i,s} = (\beta_{i1,s}, \dots, \beta_{im,s})$ is the ith row of $\mathbf{B}_s$. Setting $\mathbf{h}_t = (h_{1t}, \dots, h_{mt})'$ we assume the stationary component $\boldsymbol{h}_t$ to have the following vector GARCH (p, q) representation:

$$\mathbf{h}_t = \boldsymbol{\omega} + \sum_{r=1}^{q} \mathbf{A}_r \boldsymbol{\varepsilon}^2_{t-r} + \sum_{s=1}^{p} \mathbf{B}_s \mathbf{h}_{t-s} \tag{6}$$

where $\boldsymbol{\varepsilon}^2_t = (\varepsilon^2_{1t}, \dots, \varepsilon^2_{mt})'$ and $\boldsymbol{\omega}$ is an $m \times 1$ vector of intercepts with strictly positive entries.

To introduce time-variation in volatility dynamics, we define the nonstationarity component $\mathbf{g}_t$ as follows:

$$\mathbf{g}_t = \left(\boldsymbol{\omega}^* + \sum_{r=1}^{q} \mathbf{A}^*_r \boldsymbol{\varepsilon}^2_{t-r} + \sum_{s=1}^{p} \mathbf{B}^*_s \mathbf{h}_{t-s} \right) \boldsymbol{G}(t/T) \tag{7}$$

where $\mathbf{g}_t = (g_{1t}, \dots, g_{mt})'$ is an $m \times 1$ stochastic time-varying vector, $\boldsymbol{\omega}^*$ is an $m \times 1$ intercept vector, $\mathbf{A}^*_r = (\boldsymbol{\alpha}^{*\prime}_{1,r}, \dots, \boldsymbol{\alpha}^{*\prime}_{m,r})'$, with $\boldsymbol{\alpha}^*_{i,r} = (\alpha^*_{i1,r}, \dots, \alpha^*_{im,r})$ is the ith row of $\mathbf{A}^*_r$, and $\mathbf{B}^*_s$ are $m \times m$ matrices and $\mathbf{G}(t/T) = \mathsf{diag}(G_1(t/T; \gamma_1, \mathbf{c}_1), \dots, G_m(t/T; \gamma_m, \mathbf{c}_m))$ is a diagonal matrix of m transition functions defined below in (8). Further simplification of the model is possible by letting $\mathbf{B}_s$ and $\mathbf{B}^*_s$ to be diagonal, thereby alleviating the computational burden and reducing dimensionality without preventing the high-dimensional property of the model. In what follows, we assume $\mathbf{B}_s = \mathsf{diag}(\beta_{11,s}, \dots, \beta_{mm,s})$ and $\mathbf{B}^*_s = \mathsf{diag}(\beta^*_{11,s}, \dots, \beta^*_{mm,s})$. Each component of the matrix $\mathbf{G}(t/T)$ is represented by a general logistic transition function $G_i(t/T; \gamma_i, \mathbf{c}_i)$, $i = 1, \dots, m$, with the form

$$G_i(t/T; \gamma_i, \mathbf{c}_i) = \left(1 + \exp\left\{ -\gamma_i \prod_{k=1}^{k_i} (t/T - c_{ik}) \right\} \right)^{-1}, \gamma_i > 0, \quad c_{i1} < \dots < c_{ik_i} \tag{8}$$

where $\mathbf{c}_i = (c_{i1}, \dots, c_{ik_i})'$ is the vector of k_i location parameters. The function $G_i(t/T; \gamma_i, \mathbf{c}_i)$ is continuous for $\gamma_i < \infty$ and bounded between zero and one. By construction, when $\gamma_i = 0$ for all $i = 1, \dots, m$, the model collapses to the augmented GARCH specification of Francq and Zakoïan (2016). The parameters $\mathbf{c}_i$ and γ_i determine the location and the speed of transition between states. For small values of γ_i, the changes between volatility regimes are smooth, but when $\gamma_i \to \infty$, structural breaks can be identified at $c_{ik}, k = 1, \dots, k_i$. The dimension of $\mathbf{c}_i$ is determined by testing a sequence of nested hypotheses, as in Lin and Teräsvirta (1994) and Teräsvirta (1994). Eqs. (1)–(8) jointly define the time-varying extended conditional correlation (TV-ECC-) GARCH model. The modelling strategy for building the TV-ECC-GARCH model is similar to the specific-to-general approach for nonlinear conditional mean models discussed in, among others, Teräsvirta (1998) and Teräsvirta et al. (2010).

Furthermore, we assume that the following conditions are satisfied:

Assumption 1. The true vector of parameters $\theta_0 \in \boldsymbol{\Theta}$ lies in the interior of a compact and convex parameter space $\boldsymbol{\Theta}$.

Assumption 2. The error terms $z_{i,t} \sim$ iid with $\mathsf{E}[z_{i,t}|\mathcal{F}_{t-1}] = 0$, $\mathsf{E}[z^2_{i,t}|\mathcal{F}_{t-1}] = 1$, $i = 1, \dots, m$. In addition, $\mathsf{E}|z^{2(2+\phi)}_{i,t}| < \infty$ for some $\phi > 0$ and $i = 1, \dots, m$.

Assumption 3. In (6), the roots of $det(\mathbf{I}_m - \sum_{i=1}^{q} \mathbf{A}_i L^i - \sum_{i=1}^{p} \mathbf{B}_i L^i)$ lie outside the unit circle.

Assumption 4. The elements of $\mathbf{G}(t/T)$ satisfy $\inf_{\tau \in \Theta} G_i(t/T; \gamma_i, \boldsymbol{c}_i) \geq G_{min} > 0$, for $i = 1, \dots, m$.

Assumption 5. The slope parameters and the location parameters satisfy, respectively, the identification restrictions $\gamma_i > 0$ and $c_{i1} < \dots < c_{ik_i}$, $i = 1, \dots, m$.

Assumption 6. The matrix $\mathbf{P}$ is a finite, symmetric, and positive-definite matrix, matrix, with the elements on the main diagonal being 1 and the largest absolute eigenvalue of the matrix $\mathbf{P}$ having a positive lower bound over $\boldsymbol{\Theta}$.

Remark 1. Assumption 1 states a standard regularity condition. The "fourth-moment restriction" in Assumption 2 is necessary to ensure the existence of the score's variance. Assumption 3 presents a necessary condition for weak stationarity for the vector GARCH component. Assumption 4 is required for positivity and boundedness of the time-dependent component g_{it}, $i = 1, \dots, m$. The conditions in Assumption 5 are identification restrictions required for the existence of a unique maximum value for the log-likelihood function. Assumption 6 is needed for positive definiteness of $\boldsymbol{\Sigma}_t$.

The above formulation nests the extended constant-conditional-correlation (ECCC-) GARCH model of Jeantheau (1998) as a special case. The assumption of nonnegative volatility parameters is a sufficient condition for ensuring the positive definiteness of the conditional covariance matrix in this specification; for references, see Definition 3.1 in Jeantheau (1998), Assumption 3 in Ling and McAleer (2003), and Section 3 in Francq and Zakoïan (2012). In order to allow for negative volatility spillovers, we refer to the conditions derived in Conrad and Karanasos (2010).

Remark 2. Positive definiteness of $\boldsymbol{\Sigma}_t$ is ensured provided that the conditional variances $\sigma^2_i(t/T)$ are strictly positive and $\mathbf{P}$ is a well-behaved correlation matrix. The additional conditions that the elements of $\boldsymbol{\omega} + \boldsymbol{\omega}^*$ are positive, and the elements of $\mathbf{A}_r + \mathbf{A}^*_r$ and $\mathsf{diag}(\mathbf{B}_s + \mathbf{B}^*_s)$, $r = 1, \dots, q$, $s = 1, \dots, p$ are non-negative, are sufficient to guarantee $\sigma^2_i(t/T) > 0$, for all $i = 1, \dots, m$. When these constraints are satisfied and $\mathbf{P}$ is positive definite, the matrix $\boldsymbol{\Sigma}_t$ is positive definite almost surely for all t.

3. Equation-by-equation estimator

3.1. The log-likelihood function

We begin by introducing some notation. Let the parameter vector $\boldsymbol{\theta}$ be partitioned as $\boldsymbol{\theta} = (\boldsymbol{\vartheta}', \boldsymbol{\rho}')'$, where $\boldsymbol{\vartheta}$ denotes the volatility parameter vector and $\boldsymbol{\rho} = \mathrm{vecl}(\mathbf{P}_t)$ is the $m(m-1)/2$-dimensional vector of correlation parameters, where the operator vecl stacks the lower off-diagonal elements of the symmetric $m \times m$ matrix $\mathbf{P}_t$. The vector $\boldsymbol{\vartheta}$ is further partitioned into $\boldsymbol{\vartheta} = (\boldsymbol{\phi}', \boldsymbol{\varphi}')'$, where $\boldsymbol{\phi} = (\boldsymbol{\phi}'_1, \dots, \boldsymbol{\phi}'_m)'$ and $\boldsymbol{\varphi} = (\boldsymbol{\varphi}'_1, \dots, \boldsymbol{\varphi}'_m)'$ are vectors containing the parameters in $\mathbf{h}_t$ and $\mathbf{g}_t$, respectively. Assume that the vector $\boldsymbol{\varphi}$ is decomposed as $\boldsymbol{\varphi} = (\boldsymbol{\psi}', \boldsymbol{\tau}')'$, where $\boldsymbol{\psi} = (\boldsymbol{\psi}'_1, \dots, \boldsymbol{\psi}'_m)'$ and $\boldsymbol{\tau} = (\boldsymbol{\tau}'_1, \dots, \boldsymbol{\tau}'_m)'$. For each i, $i, j = 1, \dots, m$, denote $\boldsymbol{\phi}_i = (\omega_i, \boldsymbol{\alpha}'_{i1}, \dots, \boldsymbol{\alpha}'_{im}, \boldsymbol{\beta}'_i)'$, where $\boldsymbol{\alpha}_{ij} = (\alpha_{ij,1}, \dots, \alpha_{ij,q})'$ and $\boldsymbol{\beta}_i = (\beta_{ii,1}, \dots, \beta_{ii,p})'$, $\boldsymbol{\psi}_i = (\omega^*_i, \boldsymbol{\alpha}^{*\prime}_{i1}, \dots, \boldsymbol{\alpha}^{*\prime}_{im}, \boldsymbol{\beta}^{*\prime}_i)$, where $\boldsymbol{\alpha}^*_{ij} = (\alpha^*_{ij,1}, \dots, \alpha^*_{ij,q})'$, $\boldsymbol{\beta}^*_i = (\beta^*_{ii,1}, \dots, \beta^*_{ii,p})'$, and $\boldsymbol{\tau}_i = (\gamma_i, \boldsymbol{c}_i)'$ with $\boldsymbol{c}_i = (c_{i1}, \dots, c_{ik_i})'$. Furthermore, assume the subscript 0 denotes quantities evaluated at the true parameter values. Thus, the true parameter vector equals $\boldsymbol{\theta}_0 = (\boldsymbol{\vartheta}'_0, \boldsymbol{\rho}'_0)'$, where $\boldsymbol{\vartheta}_0 = (\boldsymbol{\phi}'_0, \boldsymbol{\varphi}'_0)'$.

Under the assumption of normality, $\boldsymbol{\varepsilon}_t|\mathcal{F}_{t-1} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_t)$, the log-likelihood function for observation t conditional on the initial values can be expressed as

$$L_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} \ell_t(\boldsymbol{\theta}) \tag{9}$$

where

$$\ell_t(\boldsymbol{\theta}) = -(m/2)\ln(2\pi) - (1/2)\ln|\boldsymbol{\Sigma}_t| - (1/2)\boldsymbol{\varepsilon}'_t \boldsymbol{\Sigma}^{-1}_t \boldsymbol{\varepsilon}_t$$

$$= -(m/2)\ln(2\pi) - (1/2)\ln|\mathbf{D}_t\mathbf{P}\mathbf{D}_t| - (1/2)\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t$$
$$= -(m/2)\ln(2\pi) - (1/2)\ln|\mathbf{D}_t|^2 - (1/2)\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t$$
$$- (1/2)\ln|\mathbf{P}| - (1/2)\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{z}_t + (1/2)\mathbf{z}_t'\mathbf{z}_t. \quad (10)$$

Maximising $L_T(\theta) = \sum_{t=1}^{T} \ell_t(\theta)$ with respect to θ yields the quasi-maximum likelihood estimator (QMLE) $\hat{\theta}_T$:

$$\hat{\theta}_T = \arg\max_{\theta_0 \in \Theta} L_T(\theta) \quad (11)$$

Eq. (10) implies the following decomposition of the log-likelihood function for observation t:

$$\ell_t(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho) = \ell_t^V(\boldsymbol{\phi}, \boldsymbol{\varphi}) + \ell_t^C(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho) \quad (12)$$

where

$$\ell_t^V(\boldsymbol{\phi}, \boldsymbol{\varphi}) = \sum_{i=1}^{m} \ell_{it}^V(\boldsymbol{\phi}_i, \boldsymbol{\varphi}_i)$$
$$= \sum_{i=1}^{m} -(1/2)\left\{\ln(2\pi) + \ln(h_{it}(\boldsymbol{\phi}_i) + g_{it}(\boldsymbol{\varphi}_i)) + \frac{\varepsilon_{it}^2}{h_{it}(\boldsymbol{\phi}_i) + g_{it}(\boldsymbol{\varphi}_i)}\right\} \quad (13)$$

and

$$\ell_t^C(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho) = -(1/2)\left\{\ln|\mathbf{P}_t(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho)| + \mathbf{z}_t'\mathbf{P}_t^{-1}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho)\mathbf{z}_t - \mathbf{z}_t'\mathbf{z}_t\right\}. \quad (14)$$

Using decomposition (12) above, maximum likelihood estimation of θ can be carried out in two steps, similar to the two-step approach suggested by Engle (2002) to estimate the DCC-GARCH model. However, due to the higher-dimensional parameter space, estimating the parameters using full maximum likelihood is computationally demanding. To alleviate the computational burden, we apply the equation-by-equation estimator proposed by Francq and Zakoïan (2016). The assumption of diagonality of $\mathbf{B}_s$ and $\mathbf{B}_s^\star$ enables separate estimation of the variance equations while still allowing for cross-market volatility interactions. This formulation facilitates the estimation of the model even when the covariance matrix is of large dimension. The equation-by-equation estimation proceeds in two steps:

1. Estimate $\boldsymbol{\phi}_i$ and $\boldsymbol{\varphi}_i$ equation-by-equation by maximising
$$L_{iT}^V(\boldsymbol{\phi}_i, \boldsymbol{\varphi}_i) = \sum_{t=1}^{T} \ell_{it}^V(\boldsymbol{\phi}_i, \boldsymbol{\varphi}_i) \quad (15)$$
with respect to $\boldsymbol{\phi}_i$ and $\boldsymbol{\varphi}_i$ for each i, $i = 1, \dots, m$, separately. This yields the estimators $\hat{\boldsymbol{\phi}}_{iT}$ and $\hat{\boldsymbol{\varphi}}_{iT}$, $i = 1, \dots, m$.
2. After estimating the volatility equations, obtain $\hat{\rho}$ given $\hat{\boldsymbol{\phi}}_{iT}$ and $\hat{\boldsymbol{\varphi}}_{iT}$ by maximising
$$L_T^C(\rho|\hat{\boldsymbol{\phi}}_T, \hat{\boldsymbol{\varphi}}_T) = \sum_{t=1}^{T} \ell_t^C(\boldsymbol{\phi}, \boldsymbol{\varphi}, \rho). \quad (16)$$

The two-step estimation of the time-varying extended conditional correlation GARCH requires careful definition of the score and the Hessian of the log-likelihood function. For space considerations, the analytical expressions of the first and second derivatives of the log-likelihood function are provided in Appendix.

3.2. Asymptotic properties

Under mild regularity conditions, Ling and McAleer (2003) established the consistency and asymptotic normality of the QMLE for the general class of vector ARMA-GARCH models (without any diagonality assumption) with constant conditional correlations. Moreover, strict stationarity and ergodicity are also proved for these models. In a restricted formulation, the consistency of the QMLE estimator for the ECCC-GARCH model was established by Jeantheau (1998) and the condition for the existence of the fourth-order moment was derived by He and Teräsvirta (2004). More recently, Francq and Zakoïan (2016) demonstrated strong consistency and asymptotic normality of the equation-by-equation estimator of the augmented GARCH model with constant conditional correlations. They showed that when errors deviate from normality, the equation-by-equation estimator remains asymptotically more efficient than the quasi-maximum likelihood approach when the parameters are jointly estimated. The asymptotic results of the second-step of the equation-by-equation estimator need yet to be further investigated.

Since under the null of constant volatility interactions, our model with constant conditional correlations collapses into the augmented GARCH model of Francq and Zakoïan (2016), we can rely on their following result. Under suitable assumptions and regularity conditions, the asymptotic distribution of the equation-by-equation estimator is

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathcal{J}_T^{-1}(\theta_0)\mathcal{I}_T(\theta_0)\mathcal{J}_T^{-1}(\theta_0)) \quad (17)$$

where $\mathcal{I}_T(\theta_0)$ and $\mathcal{J}_T(\theta_0)$ can be consistently estimated by

$$\mathcal{I}_T(\hat{\theta}_T) = (1/T)\sum_{t=1}^{T} s_t(\hat{\theta}_T)s_t(\hat{\theta}_T)' \quad (18)$$

and

$$\mathcal{J}_T(\hat{\theta}_T) = -(1/T)\sum_{t=1}^{T} \frac{\partial^2 \ell_t(\hat{\theta}_T)}{\partial\theta\partial\theta'}, \quad (19)$$

respectively. If we further assume $\mathbf{z}_t|\mathcal{F}_{t-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$, then $\mathcal{I}_T(\theta_0) = -\mathsf{E}[\mathcal{J}_T(\theta_0)]$ and the asymptotic covariance matrix reduces to $\mathcal{I}_T^{-1}(\theta_0)$.

Extending their asymptotic results to our model is nontrivial and beyond the scope of this work. For inference, we assume the asymptotic distribution of the equation-by-equation estimator to be normal. It then follows that

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathcal{I}_T^{-1}(\theta_0)). \quad (20)$$

4. Testing the adequacy of the extended vector GARCH model

In this section, we first introduce a statistical test against a general form of additive misspecification in the extended vector GARCH process and thereafter we focus our attention on the specific form of tests for parameter constancy against smooth changes in the conditional variance equations.

4.1. The general misspecification test

Over a long observation period, certain economic, environmental, and social events affecting financial markets may cause the volatility structure to change over time. Similarly, volatility transmissions between markets are likely to behave differently during tranquil and turbulent periods. It thus seems inappropriate to assume that the parameters remain constant when the series of returns to be modelled is long. Testing the constancy of parameters is therefore an important statistical tool to validate the specification of the estimated model. A rejection of the null hypothesis of parameter constancy against an additive time-dependent vector GARCH model might be seen as evidence for time-varying volatility parameters. Of course, other types of misspecification are also possible. Since the rejection of the null hypothesis does not imply that the data have been generated by the time-varying extended conditional correlation (TV-ECC-) GARCH model, the LM-type test can be viewed as a general misspecification test for multivariate GARCH models.

We shall start with the general misspecification hypothesis for which tests against specific alternatives can easily be derived, and then we present explicit formulas for the alternative of interest. In order to shorten the notation, let $h_{it} = h_{it}(\boldsymbol{\phi}_i)$ and $g_{it} = g_{it}(\boldsymbol{\varphi}_i)$, $i = 1, \dots, m$, where the additional component g_{it} is an $\mathcal{F}_{t-1}$–measurable function depending on the additional parameters $\boldsymbol{\varphi}_i$. In what follows, we assume that the true process of conditional variance is additively misspecified by introducing a new component

$$\varepsilon_{it} = \sigma_{it} z_{it}, \qquad i = 1,\dots,m \tag{21}$$

$$\sigma_{it}^2 = h_{it}(\boldsymbol{\phi}_i) + g_{it}(\boldsymbol{\varphi}_i), \tag{22}$$

where the errors z_{it} form a sequence of independent random variables with mean zero and variance one, for each $i = 1,\dots,m$. The function $g_{it} = g_{it}(\boldsymbol{\varphi}_i)$ is at least twice continuously differentiable with respect to $\boldsymbol{\varphi}_i$, such that under the null hypothesis $g_{it}(\boldsymbol{\varphi}_i) = 0$ if and only if $\boldsymbol{\varphi}_i = \mathbf{0}$. Since $\mathbf{D}_t$ is diagonal, it follows that the Gaussian quasi-log-likelihood function is further simplified to

$$\ell_{it}(\theta) = -(1/2)\{\ln 2\pi + \ln\{h_{it}(\boldsymbol{\phi}_i) + g_{it}(\boldsymbol{\varphi}_i)\} + \frac{\varepsilon_{it}^2}{h_{it}(\boldsymbol{\phi}_i) + g_{it}(\boldsymbol{\varphi}_i)}\}. \tag{23}$$

The average score of (23) is partitioned as

$$\mathbf{s}(\boldsymbol{\vartheta}_i) = (\mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\vartheta}_i)', \mathbf{s}_{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}_i)')' \tag{24}$$

where

$$\mathbf{s}(\boldsymbol{\vartheta}_i) = (1/T)\sum_{t=1}^{T} \frac{\partial \ell_{it}(\boldsymbol{\theta})}{\partial \boldsymbol{\vartheta}_i} = (2T)^{-1}\sum_{t=1}^{T}(z_{it}^2 - 1)\mathbf{x}_{it}. \tag{25}$$

The components of (25) are

$$\mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\vartheta}_i) = (1/T)\sum_{t=1}^{T} \frac{\partial \ell_{it}(\boldsymbol{\vartheta})}{\partial \boldsymbol{\phi}_i} = (2T)^{-1}\sum_{t=1}^{T}(z_{it}^2 - 1)\mathbf{x}_{1i,t} = (2T)^{-1}\sum_{t=1}^{T}(z_{it}^2 - 1)\frac{1}{h_{it} + g_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\phi}_i}, \tag{26}$$

and

$$\mathbf{s}_{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}_i) = (1/T)\sum_{t=1}^{T} \frac{\partial \ell_{it}(\boldsymbol{\theta})}{\partial \boldsymbol{\varphi}_i} = (2T)^{-1}\sum_{t=1}^{T}(z_{it}^2 - 1)\mathbf{x}_{2i,t} = (2T)^{-1}\sum_{t=1}^{T}(z_{it}^2 - 1)\frac{1}{h_{it} + g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\varphi}_i}, \tag{27}$$

where $z_{it} = \varepsilon_{it}/(h_{it} + g_{it})^{1/2}$ and $\mathbf{x}_{it} = (\mathbf{x}_{1i,t}', \mathbf{x}_{2i,t}')'$, with $\mathbf{x}_{1i,t} = (h_{it} + g_{it})^{-1}(\partial h_{it}/\partial \boldsymbol{\phi}_i)$ and $\mathbf{x}_{2i,t} = (h_{it} + g_{it})^{-1}(\partial g_{it}/\partial \boldsymbol{\varphi}_i)$.

Setting $\hat{h}_{it} = h_{it}(\hat{\boldsymbol{\phi}}_{iT})$, $\hat{g}_{it} = g_{it}(\hat{\boldsymbol{\varphi}}_{iT})$ and $\hat{z}_{it} = \varepsilon_{it}/(\hat{h}_{it} + \hat{g}_{it})^{1/2}$, the average score evaluated at $\hat{\boldsymbol{\vartheta}}_T$ under the null hypothesis yields

$$\mathbf{s}(\hat{\boldsymbol{\phi}}_{iT}, \mathbf{0}) = (\mathbf{0}', \mathbf{s}_{\boldsymbol{\varphi}}(\hat{\boldsymbol{\phi}}_{iT}, \mathbf{0})')' \tag{28}$$

where

$$\mathbf{s}_{\boldsymbol{\varphi}}(\hat{\boldsymbol{\phi}}_{iT}, \mathbf{0}) = (2T)^{-1}\sum_{t=1}^{T}(\hat{z}_{it}^2 - 1)\hat{\mathbf{x}}_{2i,t}|_{H_{0i}} \tag{29}$$

is the relevant (nonzero) block in the LM test statistic. It follows that, under regularity conditions, $\hat{\boldsymbol{\phi}}_{iT} \longrightarrow \boldsymbol{\phi}_{i0}$ and $\hat{\boldsymbol{\varphi}}_{iT} \longrightarrow \boldsymbol{\varphi}_{i0}$ in probability as $T \longrightarrow \infty$.

Denoting

$$\mathcal{I}_{iT}(\boldsymbol{\vartheta}_0) = \mathsf{E}\mathbf{s}(\boldsymbol{\phi}_{i0}, \mathbf{0})\mathbf{s}(\boldsymbol{\phi}_{i0}, \mathbf{0})' = \begin{bmatrix} \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\phi},i}(\boldsymbol{\vartheta}_0) & \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\varphi},i}(\boldsymbol{\vartheta}_0) \\ \mathcal{I}_{\boldsymbol{\varphi}\boldsymbol{\phi},i}(\boldsymbol{\vartheta}_0) & \mathcal{I}_{\boldsymbol{\varphi}\boldsymbol{\varphi},i}(\boldsymbol{\vartheta}_0) \end{bmatrix} \tag{30}$$

the corresponding south-east block of the inverse of $\mathcal{I}_{iT}(\hat{\boldsymbol{\vartheta}}_T)$ evaluated under the null equals

$$\left\{\mathcal{I}_{iT}(\hat{\boldsymbol{\vartheta}}_T)\right\}^{-1}_{[\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i]} = \left\{\mathcal{I}_{\boldsymbol{\varphi}\boldsymbol{\varphi},i}(\hat{\boldsymbol{\vartheta}}_T) - \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\varphi},i}(\hat{\boldsymbol{\vartheta}}_T)\mathcal{I}^{-1}_{\boldsymbol{\phi}\boldsymbol{\phi},i}(\hat{\boldsymbol{\vartheta}}_T)\mathcal{I}_{\boldsymbol{\varphi}\boldsymbol{\phi},i}(\hat{\boldsymbol{\vartheta}}_T)\right\}^{-1}, \tag{31}$$

where $\mathcal{I}_{\nu\kappa,i}(\hat{\boldsymbol{\vartheta}}_T)$, $\nu, \kappa = \boldsymbol{\phi}, \boldsymbol{\varphi}$, is a consistent plug-in estimator of $\mathcal{I}_{\nu\kappa,i}(\boldsymbol{\vartheta}_0)$. The subscripts ν and κ indicate the blocks defined by the block structure of $\mathcal{I}_{iT}(\boldsymbol{\vartheta}_0)$. Assuming z_{it} is normal, then

$$\mathcal{I}_{\nu\kappa,i}(\hat{\boldsymbol{\vartheta}}_T) = (2T)^{-1}\sum_{t=1}^{T}\hat{\mathbf{v}}_{\nu t,i}\hat{\mathbf{v}}_{\kappa t,i}', \quad \nu, \kappa = \boldsymbol{\phi}, \boldsymbol{\varphi}, \tag{32}$$

is a consistent estimator of $\mathcal{I}_{\nu\kappa,i}(\boldsymbol{\vartheta}_0)$ under H_0, where $\hat{\mathbf{v}}_{\nu t,i} = \hat{h}_{it}^{-1}\partial\hat{h}_{it}/\partial\nu_i$ and $\hat{\mathbf{v}}_{\kappa t,i} = \hat{g}_{it}^{-1}\partial\hat{g}_{it}/\partial\kappa_i$. Theorem 1 presents the univariate LM-type statistic for the test against a general additive alternative. Specific alternatives for the test can be easily adapted into our framework, where $\hat{z}_{it}$, $\hat{\mathbf{v}}_{\nu t,i}$, and $\hat{\mathbf{v}}_{\kappa t,i}$, $\nu, \kappa = \boldsymbol{\phi}, \boldsymbol{\varphi}$, have to be modified accordingly.

Theorem 1 (***Univariate test statistic***). *Consider the model (21)–(22) and assume that Assumptions 1–6 hold. Furthermore, assume that under the null hypothesis $H_{0i} : \boldsymbol{\varphi}_i = \mathbf{0}$, the function $g_{it} = g_{it}(\boldsymbol{\varphi}_i) \equiv 0$ and the appropriate estimates for $\mathbf{x}_{it} = (\mathbf{x}_{1i,t}', \mathbf{x}_{2i,t}')'$ are defined as $\hat{\mathbf{x}}_{1i,t} = h_{it}^{-1}(\hat{\boldsymbol{\phi}}_{iT})\frac{\partial h_{it}(\hat{\boldsymbol{\phi}}_{iT})}{\partial \boldsymbol{\phi}_i}|_{H_{0i}}$, $\hat{\mathbf{x}}_{2i,t} = h_{it}^{-1}(\hat{\boldsymbol{\varphi}}_{iT})\frac{\partial g_{it}(\hat{\boldsymbol{\varphi}}_{iT})}{\partial \boldsymbol{\varphi}_i}|_{H_{0i}}$, and $\hat{\zeta}_{it} = \varepsilon_{it}^2/h_{it}(\hat{\boldsymbol{\phi}}_{iT}) - 1$. Let $\hat{\boldsymbol{\vartheta}}_T$ be a consistent estimator of $\boldsymbol{\vartheta}_0$. Then, under the null hypothesis $H_{0i} : \boldsymbol{\varphi}_i = \mathbf{0}$, the LM type statistic for the volatility equation i*

$$\xi_{LMi} = (1/2)\sum_{t=1}^{T}\hat{\zeta}_{it}\hat{\mathbf{x}}_{2i,t}'\left\{\mathcal{I}_{iT}(\hat{\boldsymbol{\vartheta}}_T)\right\}^{-1}_{[\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i]}\sum_{t=1}^{T}\hat{\zeta}_{it}\hat{\mathbf{x}}_{2i,t} \tag{33}$$

is asymptotically χ^2-distributed with $\dim(\boldsymbol{\varphi}_i)$ degrees of freedom.

In practice, an asymptotically equivalent test to the LM test in Theorem 1 may be carried out in a straightforward way using auxiliary least squares regressions as follows:

1. Estimate $\boldsymbol{\phi}_i$ by maximum likelihood under H_{0i} and compute $\hat{\zeta}_{it} = \varepsilon_{it}^2/\hat{h}_{it} - 1$, $\hat{\mathbf{x}}_{1i,t} = \frac{1}{\hat{h}_{it}}\frac{\partial \hat{h}_{it}}{\partial \boldsymbol{\phi}_i}|_{H_{0i}}$ and $\hat{\mathbf{x}}_{2i,t} = \frac{1}{\hat{h}_{it}}\frac{\partial \hat{g}_{it}}{\partial \boldsymbol{\varphi}_i}|_{H_{0i}}$ for $t = 1,\dots,T$.
2. Regress $\hat{\zeta}_{it}$ on $\hat{\mathbf{x}}_{1i,t}$ and $\hat{\mathbf{x}}_{2i,t}$, $t = 1,\dots,T$, and obtain the coefficient of determination R_i^2.
3. Under the null hypothesis, the test statistic
$$\xi_{LMi} = T \times R_i^2 \tag{34}$$
has an asymptotic χ^2 distribution with $\dim(\boldsymbol{\varphi}_i)$ degrees of freedom.

Note that the partial derivatives $\partial\hat{h}_{it}/\partial\boldsymbol{\phi}_i|_{H_{0i}}$ and $\partial\hat{g}_{it}/\partial\boldsymbol{\varphi}_i|_{H_{0i}}$ are computed recursively, where it is assumed $\varepsilon_{i0}^2 = h_{i0} = T^{-1}\sum_{t=1}^{T}\varepsilon_{it}^2$, $i = 1,\dots,m$, as the initial values in the recursion.

To further examine whether the coefficients of the additive component in the augmented version of Eq. (22) for each $i = 1,\dots,m$, are jointly zero, we propose a multivariate version of the LM-type test statistics (33) or (34). The general procedure involves testing the joint hypothesis of no additive misspecification in the system of variance equations, so that a rejection of the null hypothesis is evidence of model misspecification. The extension of the univariate case to the multivariate case is straightforward and the multivariate test statistic is presented in Corollary 4.1.

Corollary 4.1 (***Multivariate test statistic***). *Consider the model (21)–(22) and assume that Assumptions 1–6 hold. Due to the block-diagonality of the information matrix, under the null hypothesis $H_0 : \boldsymbol{\varphi} = \mathbf{0}$, the multivariate LM-type statistic defined by*

$$\xi_{LM} = \sum_{i=1}^{m}\xi_{LMi} \tag{35}$$

where ξ_{LMi} is given by (33) or (34) for each $i = 1,\dots,m$, has an asymptotic χ^2 distribution with $\dim(\boldsymbol{\varphi})$ degrees of freedom.

The robust versions of the univariate and multivariate test statistics to non-normal innovations can be constructed using the procedure by Wooldridge (1990, 1991). The results of Wooldridge enable us to construct test statistics robust to deviations from distributional assumptions. The approach to compute the robust univariate test statistic proceeds as follows:

1. Estimate $\boldsymbol{\phi}_i$ consistently by maximum likelihood under the null hypothesis and compute $\hat{\zeta}_{it} = \varepsilon_{it}^2/\hat{h}_{it} - 1$, $\hat{\mathbf{x}}_{1i,t} = \frac{1}{\hat{h}_{it}}\frac{\partial \hat{h}_{it}}{\partial \boldsymbol{\phi}_i}|_{H_{0i}}$ and $\hat{\mathbf{x}}_{2i,t} = \frac{1}{\hat{h}_{it}}\frac{\partial \hat{g}_{it}}{\partial \boldsymbol{\varphi}_i}|_{H_{0i}}$ for $t = 1,\dots,T$.
2. Regress $\hat{\mathbf{x}}_{2i,t}$ on $\hat{\mathbf{x}}_{1i,t}$, $t = 1,\dots,T$, and save the vector of residuals $\hat{\mathbf{r}}_{it}$ from the regression.

3. Regress $\mathbf{1}_T$ on $\hat{\zeta}_{it}\hat{\mathbf{r}}_{it}$, $t = 1, \ldots, T$, and compute the residual sum of squares RSS_i. Under H_{0i}, the robust statistic

$$\xi_{robi} = T - RSS_i \tag{36}$$

has an asymptotic χ^2 distribution with $\dim(\boldsymbol{\varphi}_i)$ degrees of freedom.

For the robust multivariate statistic, repeat the previous steps 1–3, for all $i = 1, \ldots, m$, and compute the LM-type test statistic

$$\xi_{rob} = \sum_{i=1}^{m} \xi_{robi} \tag{37}$$

which is asymptotically χ^2–distributed with $\dim(\boldsymbol{\varphi})$ degrees of freedom.

4.2. Testing for smoothly time-varying volatility and spillovers

We now consider specific alternatives that belong to the general misspecification test presented in Section 4.1. We begin by deriving the test for parameter constancy against the alternative of a smoothly time-varying augmented GARCH model, and next we focus on the volatility-based statistical test of co-movements within this class of models.

In order to derive the misspecification test statistic, rewrite the variance Eq. (22) as:

$$\sigma_{it}^2 = \omega_i + \sum_{j=1}^{m} \alpha_{ij}\varepsilon_{j,t-1}^2 + \beta_i\sigma_{i,t-1}^2 + \left(\omega_i^* + \sum_{j=1}^{m} \alpha_{ij}^*\varepsilon_{j,t-1}^2 + \beta_i^*\sigma_{i,t-1}^2\right)\widetilde{G}_{it}(t/T;\gamma_i,\mathbf{c}_i) \tag{38}$$

where, for simplicity, we restrict our discussion to the case when the augmented GARCH component is of order one ($p_i = q_i = 1$) in (6)–(7) since first-order models describe well the majority of financial applications. For notational convenience, we let $\widetilde{G}_{it}(t/T;\gamma_i,\mathbf{c}_i) = G_{it}(t/T;\gamma_i,\mathbf{c}_i) - 1/2$ without losing generality.

The null hypothesis of parameter constancy against smoothly time-varying volatility corresponds to testing $H_{0i} : \gamma_i = 0$ against $H_{1i} : \gamma_i > 0$ in (38). When $\gamma_i = 0$ holds, the parameters ω_i^*, α_{ij}^*, β_i^* and $\mathbf{c}_i$, $j = 1, \ldots, m$, constitute a vector of unidentified nuisance parameters. We circumvent the identification problem by approximating $\widetilde{G}_{it}(t/T;\gamma_i,\mathbf{c}_i)$ with its first-order Taylor expansion evaluated at $\gamma_i = 0$ as in Luukkonen et al. (1988). Using Taylor's theorem, we obtain

$$\widetilde{G}_{it}(t/T;\gamma_i,\mathbf{c}_i) = \sum_{k=0}^{k_i} \gamma_i (t/T)^k \tilde{c}_{ik} + R_{it}(t/T;\gamma_i,\mathbf{c}_i) \tag{39}$$

where $R_{it}(t/T;\gamma_i,\mathbf{c}_i)$ is the remainder term. Replacing $\widetilde{G}_{it}(t/T;\gamma_i,\mathbf{c}_i)$ in (38) by (39) and rearranging the terms gives

$$\sigma_{it}^2 = \kappa_i + \sum_{j=1}^{m} a_{ij}\varepsilon_{j,t-1}^2 + b_i\sigma_{i,t-1}^2 + \sum_{k=1}^{k_i}(t/T)^k\left(\kappa_{ik}^* + \sum_{j=1}^{m} a_{ijk}^*\varepsilon_{j,t-1}^2 + b_{ik}^*\sigma_{i,t-1}^2\right) + R_{it}(t/T;\gamma_i,\mathbf{c}_i) \tag{40}$$

where the parameters are functions of the original ones in (38). Under H_{0i}, the remainder $R_{it}(t/T;\gamma_i,\mathbf{c}_i) = 0$, so that the remainder does not affect the asymptotic null distribution of the test statistic. Using (40) we can transform the original testing problem into testing against the following approximate alternative:

$$\varepsilon_{it} = \sigma_{it} z_{it}, \quad z_{it} \sim \text{iid}(0,1), \quad i = 1, \ldots, m, \tag{41}$$

where

$$\sigma_{it}^2 = \kappa_i + \sum_{j=1}^{m} a_{ij}\varepsilon_{j,t-1}^2 + b_i\sigma_{i,t-1}^2 + \sum_{k=1}^{k_i}\kappa_{ik}^*(t/T)^k + \sum_{j=1}^{m}\sum_{k=1}^{k_i} a_{ijk}^*(t/T)^k\varepsilon_{j,t-1}^2 + \sum_{k=1}^{k_i} b_{ik}^*(t/T)^k\sigma_{i,t-1}^2 \tag{42}$$

Model (41)–(42) reduces to the null model under the auxiliary null hypothesis of parameter constancy:

$$H_{0i} : \kappa_{ik}^* = a_{ijk}^* = b_{ik}^* = 0, \quad j = 1, \ldots, m, \quad k = 1, \ldots, k_i. \tag{43}$$

The following corollary defines the test statistic for testing the null hypothesis in (43).

Corollary 4.2. *Consider the model (41)–(42) and let $\boldsymbol{\phi}_i = (\kappa_i, \mathbf{a}_i', b_i)'$ with $\mathbf{a}_i = (a_{i1}, \ldots, a_{im})'$ and $\boldsymbol{\varphi}_i = (\boldsymbol{\kappa}_i^{*\prime}, \mathbf{a}_{i1}^{*\prime}, \ldots, \mathbf{a}_{im}^{*\prime}, \mathbf{b}_i^{*\prime})'$ with $\boldsymbol{\kappa}_i = (\kappa_{i1}, \ldots, \kappa_{ik_i})'$, $\mathbf{a}_{ij}^* = (a_{ij1}^*, \ldots, a_{ijk_i}^*)'$ and $\mathbf{b}_i = (b_{i1}, \ldots, b_{ik_i})'$, $i, j = 1, \ldots, m$, $k = 1, \ldots, k_i$. In addition, denote $\boldsymbol{v}_{it} = (1, \boldsymbol{\varepsilon}_{t-1}^{2\prime}, \sigma_{i,t-1}^2)'$, $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{mt})'$, $\mathbf{Z}_{1i,t} = [(t/T)^k\varepsilon_{j,t-1}^2]$ ($j = 1, \ldots, m, k = 1, \ldots, k_i$) and $\mathbf{Z}_{2i,t} = [(t/T)^k\sigma_{i,t-1}^2]$ ($i = 1, \ldots, m, k = 1, \ldots, k_i$). Under $H_0 : \boldsymbol{\varphi}_i = \mathbf{0}$, the LM-type statistic (33), where*

$$\hat{\mathbf{x}}_{i1,t} = \frac{1}{\hat{h}_{it}}\frac{\partial\hat{h}_{it}}{\partial\boldsymbol{\phi}_i}|_{H_{0i}} = \widehat{\sigma^2}_{it}^{-1}\left(\hat{\boldsymbol{v}}_{it} + b_i\frac{\partial\widehat{\sigma^2}_{i,t-1}}{\partial\boldsymbol{\phi}_i}|_{H_{0i}}\right) \tag{44}$$

$$\hat{\mathbf{x}}_{i2,t} = \frac{1}{\hat{h}_{it}}\frac{\partial\hat{g}_{it}}{\partial\boldsymbol{\varphi}_i}|_{H_{0i}} = \widehat{\sigma^2}_{it}^{-1}\left(\left((t/T), \ldots, (t/T)^{k_i}, (\text{vec}\,\mathbf{Z}_{1i,t})', (\text{vec}\,\mathbf{Z}_{2i,t})'\right)' + b_i\frac{\partial\widehat{\sigma^2}_{i,t-1}}{\partial\boldsymbol{\varphi}_i}|_{H_{0i}}\right) \tag{45}$$

is asymptotically χ^2-distributed with $\dim(\boldsymbol{\varphi}_i)$ degrees of freedom.

5. Specification strategy of additive extended GARCH models

We design a model-building cycle for the TV-ECC-GARCH model identical to the specific-to-general strategy for nonlinear models recommended by Granger (1993) and Teräsvirta (1998), among others. The technique involves a sequential procedure to specify the parameterisation of the volatility component and to determine the shape of the transition function using a sequence of LM-type tests. The modelling cycle for specifying additive extended GARCH models consists of the following stages:

1. Estimate the extended GARCH model as in Francq and Zakoïan (2016). The determination of the lag structure may be done using a model selection criterion. This may be preceded by testing the null hypothesis of no volatility interactions as in Nakatani and Teräsvirta (2009) and Pedersen (2017).
2. Test parameter constancy against the additive time-dependent vector GARCH model (TV-ECC-GARCH) alternative using the LM-type statistic described in Section 4.2 at the significance level $\alpha^{(1)}$: $H_{0i}^{\text{TV-ECC}} : \kappa_{ik}^* = a_{ijk}^* = b_{ik}^* = 0, \quad j = 1, \ldots, m, \quad k = 1, \ldots, k_i$. in (36) in Section 4.2. If $H_{0i}^{\text{TV-ECC}}$ is rejected, then select the order $k \le 3$ in the exponent of $G_{it}(t/T;\gamma_i,\mathbf{c}_i)$ based on a sequence of nested tests as in Teräsvirta (1994). This is done by testing the following sequence of nested hypotheses:

   $$H_{03i} : \boldsymbol{\varphi}_{i3} = \mathbf{0}$$
   $$H_{02i} : \boldsymbol{\varphi}_{i2} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{i3} = \mathbf{0}$$
   $$H_{01i} : \boldsymbol{\varphi}_{i1} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{i2} = \boldsymbol{\varphi}_{i3} = \mathbf{0}$$

   where $\boldsymbol{\varphi}_{ik} = (\kappa_{ik}^*, a_{i1k}^*, \ldots, a_{imk}^*, b_{ik}^*)'$, $i = 1, \ldots, m, k = 1, 2, 3$, using LM-type tests and auxiliary regressions. The choice of k proceeds as follows. Carry out the three sequential tests and observe the hypotheses rejected. If H_{01i} and H_{03i} are rejected more strongly, as measured by the p-values, than H_{02i}, then select $k = 1$ or $k = 3$. If testing H_{02i} returns the strongest rejection, then select $k = 2$.
3. If parameter constancy is rejected, then sub-hypotheses are tested to determine whether the TV-ECC-GARCH model is necessary to characterise the data or a model with a subset of time-varying parameters is sufficient. These statistical tests are conducted using the significance level $\alpha^{(2)} = \tau\alpha^{(1)}$, where $\tau \in$

(0, 1). Here we set $\tau = 0.5$. In what follows, assume the partitioned vector $\boldsymbol{\varphi}_{ik} = (\boldsymbol{\varphi}'_{ik,1}, \boldsymbol{\varphi}'_{ik,2})'$, where $\boldsymbol{\varphi}_{ik,1} = (\kappa^*_{ik}, a^*_{ik}, b^*_{ik})'$ denotes the vector of standard GARCH parameters and $\boldsymbol{\varphi}_{ik,2} = \{a^*_{ijk}\}$, $i \neq j, j = 1, \ldots, m$, $k = 1, 2, 3$, denotes the cross-market ARCH coefficients. In order to identify individual changes in the dynamics of volatility (or of volatility co-dependence) for a subset of return series, we shall proceed as follows:

(a) Test the hypothesis of parameter constancy in the standard GARCH coefficients

$$H^{\text{TV-VOL}}_{0i,1} : \boldsymbol{\varphi}_{i1,1} = \boldsymbol{\varphi}_{i2,1} = \boldsymbol{\varphi}_{i3,1} = \mathbf{0}_3 \quad | \quad \boldsymbol{\varphi}_{i1,2} = \boldsymbol{\varphi}_{i2,2} = \boldsymbol{\varphi}_{i3,2} = \mathbf{0}.$$

(b) Test the hypothesis of parameter constancy in the cross-market ARCH coefficients

$$H^{\text{TV-CO-VOL}}_{0i,2} : \boldsymbol{\varphi}_{i1,2} = \boldsymbol{\varphi}_{i2,2} = \boldsymbol{\varphi}_{i3,2} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{i1,1} = \boldsymbol{\varphi}_{i2,1} = \boldsymbol{\varphi}_{i3,1} = \mathbf{0}.$$

(c) If $H^{\text{TV-VOL}}_{0i,1}$ or $H^{\text{TV-CO-VOL}}_{0i,2}$ is rejected, then select $k = 1, 2, 3$, by testing the following sequence of nested hypotheses at the significance level $\alpha^{(3)} = \tau\alpha^{(2)}$:

$$H_{03i,s} : \boldsymbol{\varphi}_{i3,s} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{ik,r} = \mathbf{0}$$
$$H_{02i,s} : \boldsymbol{\varphi}_{i2,s} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{ik,r} = \mathbf{0} \text{ and } \boldsymbol{\varphi}_{i3,s} = \mathbf{0}$$
$$H_{01i,s} : \boldsymbol{\varphi}_{i1,s} = \mathbf{0} \quad | \quad \boldsymbol{\varphi}_{ik,r} = \mathbf{0} \text{ and } \boldsymbol{\varphi}_{i3,s} = \boldsymbol{\varphi}_{i2,s} = \mathbf{0},$$

for $r \neq s$, $r, s = 1, 2$, using auxiliary regressions as before. Rejection of $H_{0i,2}$ provides evidence for time-varying cross-market ARCH effects, revealing changes in volatility spillovers among asset returns.

An appealing feature of this testing approach is that it makes it possible to identify individual changes in the volatility dynamics (or volatility co-dependence) for a subset of return series. As a result, statistical evidence of time-varying volatility parameters is a necessary, albeit not sufficient, condition for volatility-based spillovers. In this framework, rejecting the null hypothesis suggests that the volatility parameters are jointly time-varying, but it does not imply that every coefficient is changing over time. As a matter of fact, the hypothesis for parameter constancy can be rejected when structural changes occur in the volatility dynamics, but the co-volatility parameters remain constant.

6. Small sample properties

6.1. Design of the experiments

In this section, we investigate by Monte Carlo simulations the finite sample behaviour of the tests presented in Section 4. We shall first report the results on the empirical size and power of the parameter constancy tests. We also perform several robustness checks on the implementation of the tests. The experiments are conducted for the bivariate, trivariate and five-variate cases, that is, for $m = 2, 3, 5$. We generated 5000 replications at sample sizes $T = 1000, 2500$ and 5000 for each data generating process (DGP). The first 1000 generated observations from each data set were discarded to reduce initialisation effects. To investigate the size of the test, we simulate data from the bivariate extended constant CC-GARCH model of Jeantheau (1998) whose volatility component is given by

$$\mathbf{h}_t = \boldsymbol{\omega} + \mathbf{A}_1 \boldsymbol{\varepsilon}^2_{t-1} + \mathbf{B}_1 \mathbf{h}_{t-1} \tag{46}$$

where we restrict the matrix of GARCH parameters $\mathbf{B}_1$ to be diagonal and we let the conditional correlation parameter ρ_{12} vary between 0.3 and 0.9. The generated data satisfy the weak stationarity condition established in Jeantheau (1998) where the spectral radius of $\mathbf{A} + \mathbf{B}$, denoted by $\lambda(\mathbf{A} + \mathbf{B})$, is smaller than unity. To study the power of the test, we simulate data from the extended version of (46) with time-varying parameter matrices where the model is specified as

$$\mathbf{h}_t = \boldsymbol{\omega} + \mathbf{A}_1 \boldsymbol{\varepsilon}^2_{t-1} + \mathbf{B}_1 \mathbf{h}_{t-1} + \left(\boldsymbol{\omega}^* + \mathbf{A}^*_1 \boldsymbol{\varepsilon}^2_{t-1} + \mathbf{B}^*_1 \mathbf{h}_{t-1}\right) \mathbf{G}(t/T) \tag{47}$$

where $\mathbf{A}$ and $\mathbf{A}^*$ are non-diagonal matrices of ARCH coefficients, $\mathbf{B}$ and $\mathbf{B}^*$ are diagonal matrices of GARCH coefficients, and $\mathbf{G}(t/T)$ is the diagonal matrix of logistic transition functions defined in (8). Only parameter combinations satisfying the sufficient conditions for weak stationary in each extreme volatility state are considered. All computations have been performed using the open-source statistical software R.

Table 1
Data generating processes for size simulations for tests of parameter constancy.

	DGP 1	DGP 2	DGP 3	DGP 4
$\mathbf{A}_1$	$\begin{bmatrix} 0.10 & 0.005 \\ 0.005 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 0.10 & 0.005 \\ 0.005 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 0.10 & 0.07 \\ 0.02 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 0.10 & 0.005 \\ 0.005 & 0.05 \end{bmatrix}$
$\mathbf{B}_1$	$\begin{bmatrix} 0.80 & 0 \\ 0 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 0.80 & 0 \\ 0 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 0.80 & 0 \\ 0 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 0.88 & 0 \\ 0 & 0.94 \end{bmatrix}$
ρ_{12}	0.90	0.30	0.90	0.30
$\lambda(\mathbf{A}_1 + \mathbf{B}_1)$	0.905	0.905	0.937	0.992

Note: The vector of constants in the conditional variances equals $\boldsymbol{\omega} = [0.10 \quad 0.20]'$ for all DGPs.

6.2. Size simulations

Having investigated the statistical properties of the parameter constancy test in small samples, we first present the results from the size simulations. The data generated processes (DGPs) from four bivariate ECC-GARCH(1,1) models used in the size simulations are reported in Table 1. The artificial series have the following dynamics. The persistence in volatility varies from moderate (0.905) in DGP 1–2 to very high (0.992) in DGP 4. DGPs 2 and 4 have low correlations ($\rho_{12} = 0.3$) with moderate and very high persistence, respectively, while DGP 3 is characterised by higher cross-market volatility interactions and high correlations ($\rho_{12} = 0.9$). The simulation study shows that the non-robust tests are severely size distorted and the robust statistics clearly outperform their non-robust versions. These simulations are available on request. For the sake of saving space, we only report the actual rejection frequencies for the robust test statistics when $T = 2500, 5000$.

The size discrepancy plots for the robust tests statistics are shown in Fig. 1. The size discrepancy is the difference between the actual rejection frequency and the nominal size (vertical axis) plotted against the nominal size (horizontal axis). As expected, the size distortions decrease with the number of observations, but increase with the level of persistence in volatility. Furthermore, size distortions tend to increase with the level of correlation and decrease with the magnitude of volatility interactions. Overall, the robust test has good size in finite samples, with the exception for DGP1 characterised by high correlation and small volatility interactions, for which the test seems largely size distorted. Thus, the test may tend to reject more often when the correlations are large and the volatility interactions are small. Looking at the results of the multivariate test, the empirical size is also distorted to some extent. The size properties of the tests suggest that the asymptotic distribution of the test statistics may be a poor approximation of their true distribution in some situations. In order to improve inference and obtain well-behaved tests, bootstrap procedures may be used in order to mitigate the poor size properties in those cases.

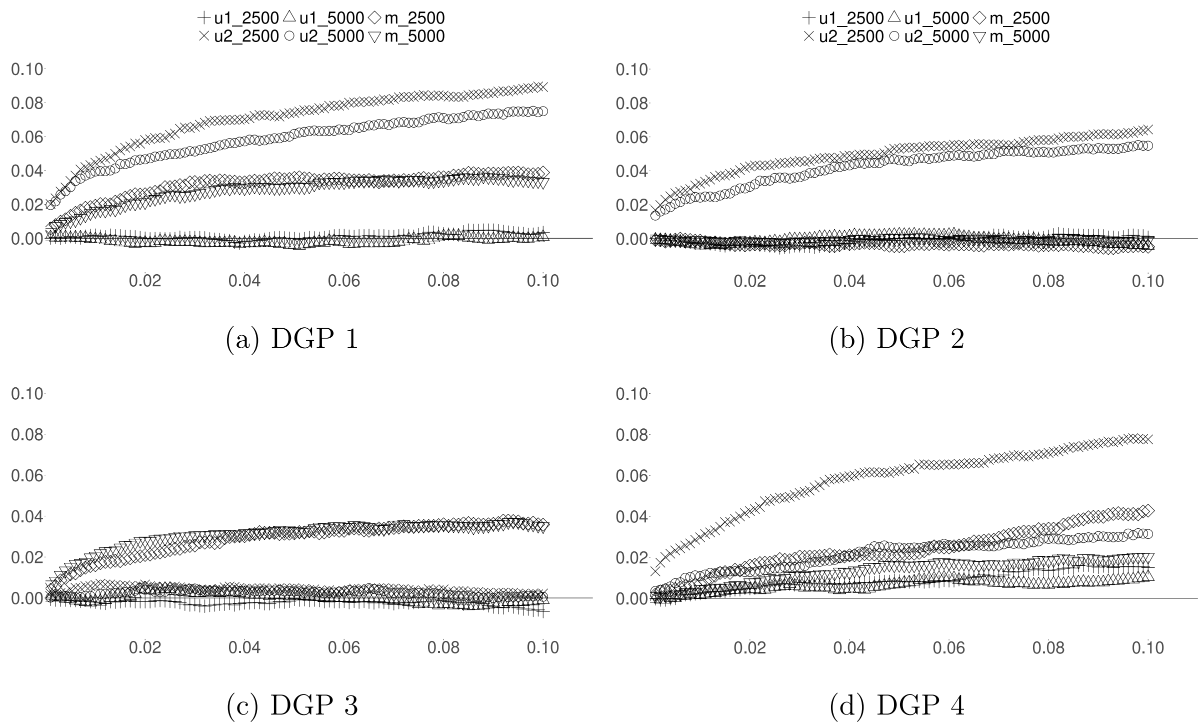


Fig. 1. Size discrepancy plots for tests of parameter constancy tests using artificial series generated according to the DGPs in Table 1. The size discrepancy is plotted against the nominal size. Results are shown for the univariate robust (ui_T), $i = 1, 2$, and multivariate robust (m_T) test statistics defined in (36) and (37), respectively, with $\hat{\mathbf{x}}_{i1,t}$ and $\hat{\mathbf{x}}_{i2,t}$ given in (44) and (45) for $T = 2500, 5000$. The number of replications for each simulation equals 5000.

6.3. Power simulations

In this section, we present the results from the power simulations for the robust test statistics. There is no direct benchmark at which we can compare our tests, but it may be interesting to investigate their power in small samples. We simulate the bivariate model under the alternative using four alternative specifications. The DGPs from (47) are listed in Table 2. The upper panel defines the volatility structure in the first extreme state, common to all DGPs, and from where parameters change to the second extreme volatility state. The sign and magnitude of the parameters that change over time are shown in the lower panel. In DGPs 5 and 6 the ARCH and GARCH parameter matrices are time-varying, respectively. DGPs 7 and 8 have a similar design, but the vector of constants is also assumed to be time-dependent. The results of the power simulations using selected models are shown in Fig. 2. The rejection frequencies for the tests under the generated data behave as expected. The actual rejection frequencies show that power increases, most notably, with the sample size, meaning that a long time series is required for a well-behaved test in terms of power. Detecting time dependence in the volatility interactions can be quite difficult against DGP 5, but the power increases sharply if the true alternative contains additional changes in the constants at all sample sizes. Interestingly, the tests also become less powerful against DGPs when the unconditional correlations are large. In general, the power against DGPs clearly increases with the number of time-varying parameters. Therefore, one expects the power of the multivariate test to be clearly stronger compared to that of the univariate parameter constancy test.

7. Empirical application

In this section, we illustrate the usefulness of the multivariate time-varying extended GARCH framework to sovereign bond markets. We use daily data on 10-year government bond yields to investigate the volatility spillovers between sovereign bonds markets from Greece (GR), Ireland (IR) and Portugal (PT) from October 3, 2005 until September 30, 2015 (2608 observations). The data were collected from the Thomson Reuters Datastream and transformed into continuously compounded rates of return in order to handle nonstationarity in the mean. To avoid estimation problems, the observations in the series are truncated with a threshold of $+/-10$ standard deviations above/below extremely large returns. The aim is to investigate cross-market volatility transmissions and the dynamics of bond market co-movements during the global financial crisis and the European sovereign debt crisis. The daily 10-year government benchmark bond yields and their truncated returns are depicted in Fig. 3. After the Greek deficit revision, yields of Greek government bonds rose sharply, followed by Portugal and Ireland. In fact, the Greek financial bailout in May 2010 was followed by the Irish bailout in November 2010 and the Portuguese bailout in April 2011. A closer look at the percent changes reveals an increasing pattern in the volatility on these European sovereign bond markets starting in late 2009. For the Irish series, the increase in volatility is more pronounced at the end of the sample period.

Summary statistics for the truncated series of yield percent changes can be found in Table 3. As expected, the bond yield returns exhibit a skewed and heavy-tailed distribution suggesting the series might not be normally distributed. The values of the robust portmanteau Q(5) statistic of Francq and Zakoïan (2009) show time-dependence in the first moment of the Greek and Portuguese bond returns. To filter out this linear dependence, we fit an AR(1) to the Greek bond returns and an ARMA(1,1) to the Portuguese bond returns. There is also evidence of higher-order ARCH effects in the bond returns from the results of the LM test of Engle (1982).

We apply the modelling strategy for the additive extended GARCH model and begin with the appropriate LM-type tests. First, we investigate the hypothesis that the ARCH and GARCH matrices are diagonal. The tests proposed by Nakatani and Teräsvirta (2009) and Pedersen (2017) are performed under the null hypothesis of no volatility spillovers. The results (not shown here but available upon request) strongly indicate the presence of volatility interactions as the p-values of the test statistics are smaller than 0.05. Next, we turn our attention

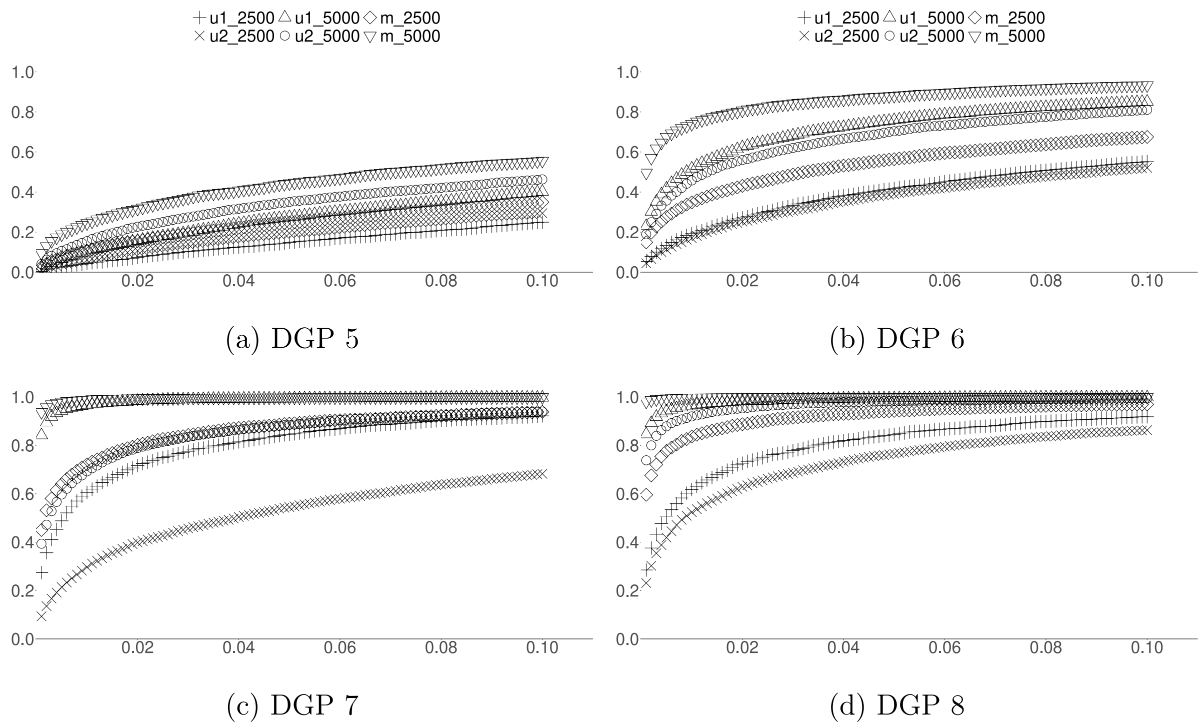


Fig. 2. Power curves of the tests for parameter constancy tests using artificial series generated according to the DGPs in Table 2. Results are shown for the univariate robust (ui_T), $i = 1, 2$, and multivariate robust (m_T) test statistics defined in (36) and (37), respectively, with $\hat{\mathbf{x}}_{i1,t}$ and $\hat{\mathbf{x}}_{i2,t}$ given in (44) and (45) for $T = 2500, 5000$. The number of replications for each simulation equals 5000.

Table 2
Data generating processes for power simulations for tests of parameter constancy.

$$\boldsymbol{\omega} = \begin{bmatrix} 0.10 & 0.20 \end{bmatrix}' \quad \mathbf{A}_1 = \begin{bmatrix} 0.10 & 0.05 \\ 0.05 & 0.05 \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} 0.80 & 0 \\ 0 & 0.85 \end{bmatrix}$$

	DGP 5	DGP 6	DGP 7	DGP 8
$\boldsymbol{\omega}^*$			$\begin{bmatrix} 0.20 & 0.10 \end{bmatrix}'$	$\begin{bmatrix} 0.20 & 0.10 \end{bmatrix}'$
$\mathbf{A}_1^*$	$\begin{bmatrix} 0.01 & 0.015 \\ 0.015 & 0.01 \end{bmatrix}$		$\begin{bmatrix} 0.01 & 0.015 \\ 0.015 & 0.01 \end{bmatrix}$	
$\mathbf{B}_1^*$		$\begin{bmatrix} 0.04 & 0 \\ 0 & 0.03 \end{bmatrix}$		$\begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}$
$\lambda\left(\mathbf{A}_1^{(*)} + \mathbf{B}_1^{(*)}\right)$	0.995	0.975	0.995	0.975

Note: In the power simulations, we set $c_{11} = 0.50$, $c_{21} = 0.75$, $\gamma_1 = 5$, $\gamma_2 = 10$ and $\rho = 0.70$ for all DGPs.

Table 3
Descriptive statistics of the daily government bond yield percent changes.

	MIN.	MEAN	MAX.	S.D.	SK	KR	Q(5)	ARCH(5)
GREECE	−27.95	0.085	27.95	2.606	−0.592	28.96	16.28	40.08
					(0.000)	(0.000)	(0.006)	(0.000)
IRELAND	−19.09	−0.019	19.09	1.914	1.012	19.84	10.46	79.32
					(0.000)	(0.000)	(0.063)	(0.000)
PORTUGAL	−20.44	0.011	18.46	2.022	0.526	12.48	19.84	18.50
					(0.000)	(0.000)	(0.001)	(0.000)

Note: SK and KR denote, respectively, the excess kurtosis and skewness. Q(5) is the portmanteau test statistic for serial correlation of Francq and Zakoïan (2009) robust to the presence of ARCH effects up to order 5 and ARCH(5) is the test for ARCH effects of Engle (1982) up to order 5 (p-values in parentheses).

to testing parameter constancy against the extended additive time-dependent vector GARCH model. Robust versions of the univariate and multivariate test statistics of constant volatility interactions are reported in Table 4. The results of the test statistics for the constancy of the full set of volatility parameters are shown in the upper panel. It is seen from the p-values that the null of constant parameters is strongly rejected by the univariate and multivariate test statistics, suggesting that the volatility parameters are time-varying, thereby providing support for the TV-ECC-GARCH model. In what follows, we attempt to disentangle changes in the dynamics of volatility (or co-volatility) by showing the parameter constancy test results in the standard GARCH coefficients and cross-market ARCH effects in the middle and lower panels of Table 4, respectively. The strong rejection of the parameter constancy hypothesis suggests structural changes in the standard GARCH coefficients and co-volatility processes during the observation period.

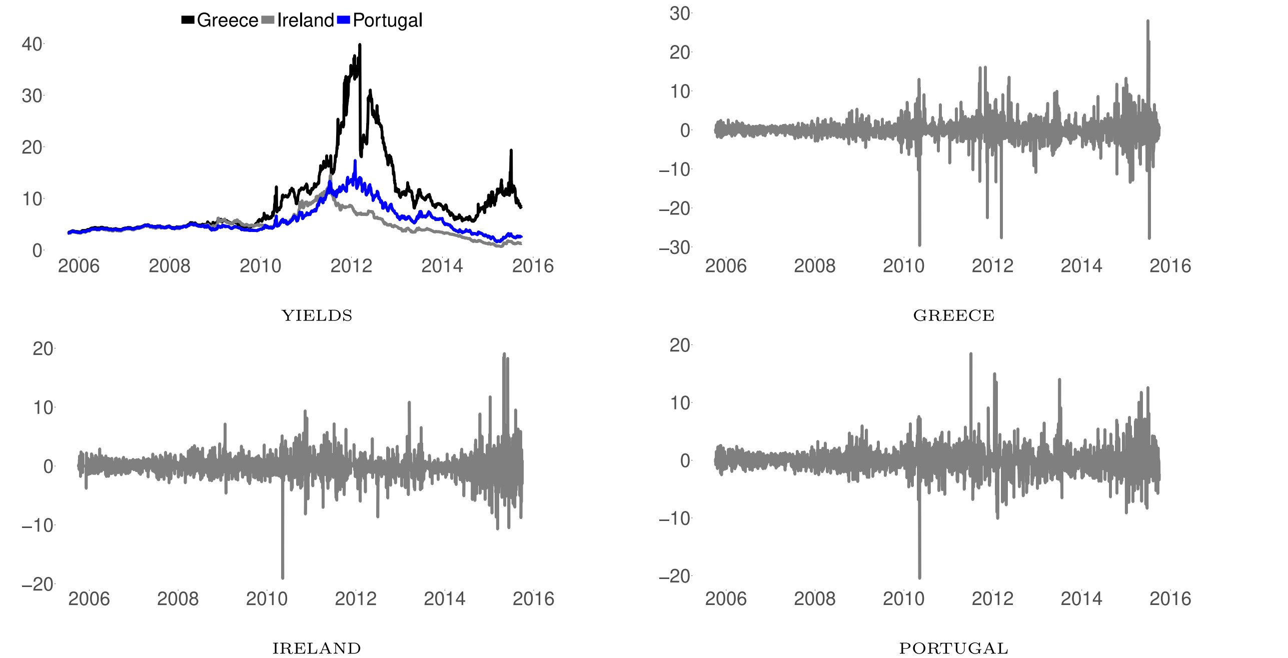


Fig. 3. Daily 10-year government bond yields and truncated percent changes. The threshold for truncation is +/− 10 times the sample standard deviation.

Table 4
Robust test statistics from the testing for parameter constancy in the univariate and multivariate forms for all parameters (upper panel), for the standard GARCH coefficients (middle panel) and for the cross-market ARCH coefficients (lower panel). The *p*-values are reported in parentheses.

	GREECE	IRELAND	PORTUGAL
H_{0i}: $\boldsymbol{\varphi}_{i3}=\boldsymbol{\varphi}_{i2}=\boldsymbol{\varphi}_{i1}=\mathbf{0}_5$	33.72	40.35	29.83
	(0.004)	(0.000)	(0.013)
H_{03i}: $\boldsymbol{\varphi}_{i3}=\mathbf{0}_5$	8.043	3.988	6.281
	(0.154)	(0.551)	(0.280)
H_{02i}: $\boldsymbol{\varphi}_{i2}=\mathbf{0}_5 \mid \boldsymbol{\varphi}_{i3}=\mathbf{0}_5$	2.618	17.53	1.962
	(0.759)	(0.004)	(0.854)
H_{01i}: $\boldsymbol{\varphi}_{i1}=\mathbf{0}_5 \mid \boldsymbol{\varphi}_{i3}=\boldsymbol{\varphi}_{i2}=\mathbf{0}_5$	13.32	7.102	15.64
	(0.021)	(0.213)	(0.008)
H_0: $\boldsymbol{\varphi}_1=(\boldsymbol{\varphi}'_{GR,1},\boldsymbol{\varphi}'_{IR,1},\boldsymbol{\varphi}'_{PT,1})'=\mathbf{0}_{15}$		36.06	
		(0.002)	
$H_{0i,1}$: $\boldsymbol{\varphi}_{i3,1}=\boldsymbol{\varphi}_{i2,1}=\boldsymbol{\varphi}_{i1,1}=\mathbf{0}_3$	30.35	25.33	15.85
	(0.000)	(0.003)	(0.051)
$H_{03i,1}$: $\boldsymbol{\varphi}_{i3,1}=\mathbf{0}_3$	5.467	3.256	3.473
	(0.141)	(0.354)	(0.324)
$H_{02i,1}$: $\boldsymbol{\varphi}_{i2,1}=\mathbf{0}_3 \mid \boldsymbol{\varphi}_{i3,1}=\mathbf{0}_3$	1.327	11.76	1.388
	(0.723)	(0.008)	(0.708)
$H_{01i,1}$: $\boldsymbol{\varphi}_{i1,1}=\mathbf{0}_3 \mid \boldsymbol{\varphi}_{i3,1}=\boldsymbol{\varphi}_{i2,1}=\mathbf{0}_3$	9.396	6.926	11.56
	(0.024)	(0.074)	(0.009)
$H_{0i,2}$: $\boldsymbol{\varphi}_{i3,2}=\boldsymbol{\varphi}_{i2,2}=\boldsymbol{\varphi}_{i1,2}=\mathbf{0}_2$	24.66	9.287	19.27
	(0.000)	(0.158)	(0.004)
$H_{03i,2}$: $\boldsymbol{\varphi}_{i3,2}=\mathbf{0}_2$	3.236	3.361	1.406
	(0.198)	(0.186)	(0.495)
$H_{02i,2}$: $\boldsymbol{\varphi}_{i2,2}=\mathbf{0}_2 \mid \boldsymbol{\varphi}_{i3,2}=\mathbf{0}_2$	4.304	6.338	6.446
	(0.116)	(0.042)	(0.040)
$H_{01i,2}$: $\boldsymbol{\varphi}_{i1,2}=\mathbf{0}_2 \mid \boldsymbol{\varphi}_{i3,2}=\boldsymbol{\varphi}_{i2,2}=\mathbf{0}_2$	8.784	0.735	3.849
	(0.012)	(0.692)	(0.146)

The specification of the TV-ECC-GARCH model includes determining the parameterisation of the transition function by choosing k_i. Upon rejecting the hypothesis of parameter constancy, we attempt to identify the number of locations using the sequence of nested null hypothesis as discussed in Section 5. Since the null hypothesis which yields the strongest rejection (measured by the *p*-value of the robust test) is H_{02i}, the rule is to select $k_i = 2$ for the Irish bond market. For the Greek and Portuguese markets, a single transition parameter suffices to capture the dominant changes in the volatility dynamics, since the *p*-value of the test of H_{01i} is the smallest. For a more detailed discussion of this procedure, see Teräsvirta (1994).

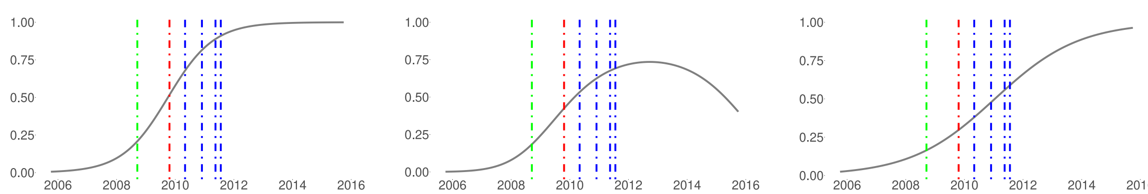
For fitting the TV-ECC-GARCH model, we use the two-step approach of Francq and Zakoïan (2016). In that setting, the algorithm consists of first estimating the individual conditional variances equation by equation and obtaining estimates of the conditional correlations in the second step. To obtain fully efficient estimates, the procedure must be repeated iteratively until convergence, where the estimates are obtained by maximising the full log-likelihood function. Compared to the two-step procedure, the multi-step approach yields more accurate estimates, but the computational burden is severely aggravated and the estimation problem becomes numerically more difficult. The two-step estimates of the model can be found in Table 5. The estimation results are broadly consistent with the results of the parameter constancy tests. The estimates shown in the upper panel of Table 5 indicate the presence of structural changes in the volatility dynamics. We find an increasing trend in the baseline volatility for the three bond markets. Not surprisingly, the level of persistence increases from the low- to the high-volatility extreme state over time. This is precisely what one would expect when moving from a tranquil to a turbulent period in the bond markets. This change in volatility persistence for Ireland and Portugal is explained by higher ARCH effects and smaller GARCH coefficients.

The off-diagonal elements of the estimated ARCH matrix for each extreme volatility state are reported in Table 5. The results suggest that the bi-directional cross-market volatility effects between the Portuguese and Greek sovereign bonds changed over time. In other words, the effect of shocks to the Portuguese bond market on the volatility of Greek bonds increased significantly after October 2009. Conversely, we observe that the effect of shocks to the Greek bond on the volatility of the Portuguese bond is notably larger before January 2011. The effects on the volatility of the Irish bond market from shocks to the Greek bond seem to be stronger before March 2010 and after April 2015. Shocks to the Portuguese bond have an impact on the volatility of Irish bonds that is significantly higher between March 2010 and

Table 5
Estimation results (standard errors in parentheses) for the variance equations (upper panel) and transition functions (lower panel) of the TV-ECC-GARCH(1,1) model in the extreme states.

	STATE I				STATE II		
	GREECE	IRELAND	PORTUGAL		GREECE	IRELAND	PORTUGAL
$\hat{\boldsymbol{\omega}}$	0.117 (0.042)	0.012 (0.007)	0.001 (0.008)	$\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{\omega}}^*$	0.609 (0.059)	0.053 (0.036)	0.131 (0.043)
	0.091 (0.034)	0.006 (0.015)	0.000 (0.028)		0.243 (0.013)	0.000 (0.006)	0.015 (0.007)
$\hat{\mathbf{A}}_1$	0.023 (0.011)	0.000 (0.017)	0.013 (0.016)	$\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_1^*$	0.000 (0.013)	0.211 (0.149)	0.034 (0.014)
	0.040 (0.011)	0.000 (0.008)	0.008 (0.015)		0.002 (0.004)	0.020 (0.008)	0.032 (0.009)
	0.702 (0.080)				0.742 (0.011)		
$\hat{\mathbf{B}}_1$		0.947 (0.016)		$\hat{\mathbf{B}}_1 + \hat{\mathbf{B}}_1^*$		0.755 (0.133)	
			0.944 (0.012)				0.924 (0.014)

	GREECE	IRELAND	PORTUGAL
$\hat{\gamma}_i$	12.76 (1.773)	15.49 (12.91)	6.962 (2.233)
$\hat{c}_{i1}$	0.400 (0.032) Oct 2009	0.441 (0.205) Mar 2010	0.529 (0.084) Jan 2011
$\hat{c}_{i2}$		0.954 (0.178) Apr 2015	



(a) GREECE (b) IRELAND (c) PORTUGAL

Fig. 4. Estimated transition functions for the Greek, Irish and Portuguese sovereign bond returns. The dashed vertical lines indicate the dates of the Lehman Brothers bankruptcy (green line), the Greek deficit revision (red line) and the first Greek financial bailout in May 2010, the Irish bailout in November 2010, the Portuguese bailout in May 2011 and the second Greek bailout in July 2011 (blue line).

April 2015. Our results also suggest higher effects on the volatility of the Portuguese sovereign bond market after January 2011 from shocks to the Irish bond. Overall, volatility interactions seem stronger during the most acute phase of the European sovereign debt crisis. Our findings suggest that volatility-based contagion is bi-directional for the pair Greece–Portugal and Ireland –Portugal and uni-directional from Greece to Ireland.

In the lower panel of Table 5 are reported the estimated transition parameters. The slope estimates exhibit smooth changes between the volatility extreme states as depicted in Fig. 4. From the results, a single transition seems to be sufficient to capture the dominant changes in the volatilities of the Greek and Portuguese markets, whereas two major structural changes can be identified in the Irish bond market. By looking at the locations of transition, the mid-point of change[2] occurs in October 2009 for the Greek sovereign bond market and in January 2011 for the Portuguese sovereign bond, few months before its financial request. With respect to the Irish sovereign bond, the first and second structural changes occur, respectively, in March 2010 and April 2015.

A closer look at the estimated volatilities of the sovereign bond returns in Fig. 5 shows an increasing trend following the Greek deficit revision (marked by the red dotted line), which is particularly remarkable for Ireland and Portugal towards the end of the sample period. The results further suggest that, when the ARCH effects are assumed to be constant, the impact of larger shocks tends to be underestimated by the ECCC-GARCH model. Therefore, constant volatility interactions may be insufficient to capture all the variation in daily volatilities during periods of market distress. This result is especially notable for the volatility processes of Greek and Irish sovereign bonds. For smaller shocks and calm periods, the estimated volatility processes from the TV-ECC-GARCH and ECCC-GARCH models remain very close. Extending the specification by allowing cross effects between markets does improve the model fit to the data according to model selection by the Bayesian information criterion. Accounting for nonstationarity in

[2] For a general transition function, the transition may be already half-completed at c if γ is small. When the speed of transition is very high, it has no implication for defining the phases of transition using the estimated locations.

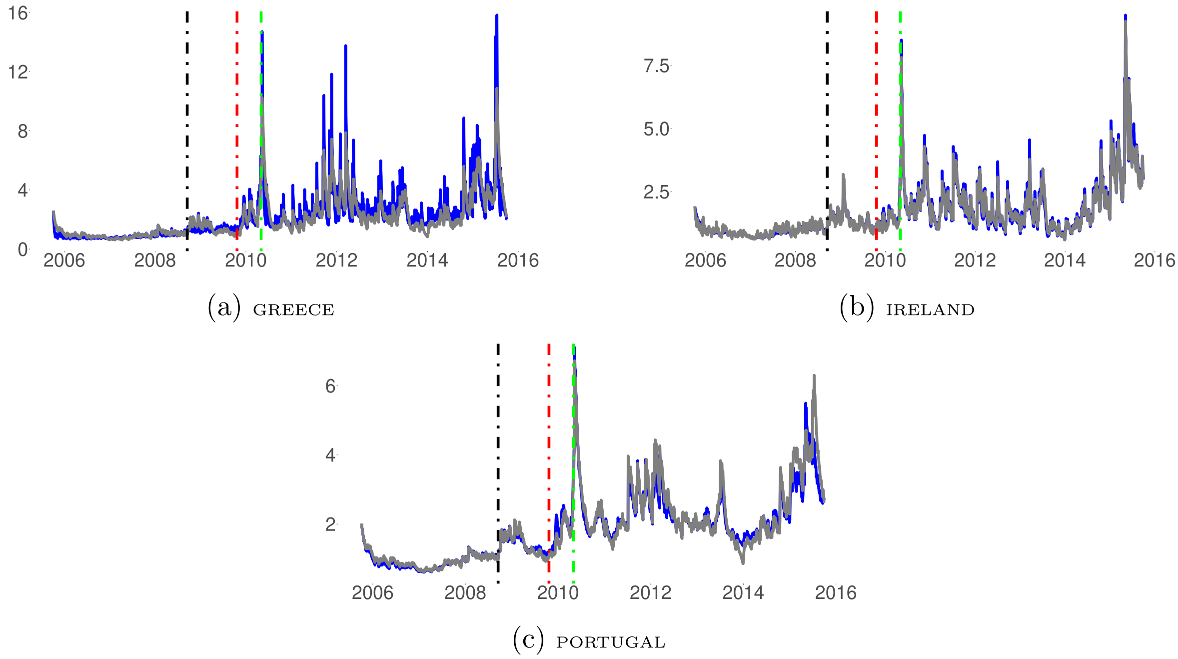


Fig. 5. Estimated conditional volatilities for the Greek, Irish and Portuguese sovereign bonds from the TV-ECC-GARCH(1,1) model (blue curve) and ECCC-GARCH(1,1) model (grey curve). The dashed vertical lines indicate the dates of the Lehman Brothers bankruptcy (black line), the Greek deficit revision (red line) and the first Greek financial bailout (green line).

the volatility equations leads to further improvement in the estimation of the model as persistence is notably reduced and the information criteria are minimised by fitting the TV-ECC-GARCH model. Following the suggestion of two reviewers, an out-of-sample validation exercise has also been implemented, albeit the results are not presented here. Preliminary results show that the model with time-varying volatility interactions has superior predictive ability compared to the benchmark GARCH model using the mean squared error and quasi-likelihood loss functions.

Different forms of parametrisations are possible for the correlation structure. The test results associated with the hypothesis of constant conditional correlations strongly reject the null hypothesis against the alternative of dynamic conditional correlations of Engle (2002) and smoothly time-varying correlations of Silvennoinen and Teräsvirta (2015). These results are not shown, but they are available upon request. For more details on the tests, we refer to Engle (2002) and Silvennoinen and Teräsvirta (2005, 2015). In what follows, we allow the unconditional correlations to change smoothly between two extreme states as in the smooth transition conditional correlation model of Silvennoinen and Teräsvirta (2015). Assuming the time-varying correlations change deterministically, the model is called TV-ETVC-GARCH(1,1) model. For comparison, we also plot the estimated (short-term) correlation structure of Engle (2002) (from the model we call TV-EDCC-GARCH) and the constant correlations obtained from the TV-ECCC-GARCH model. The estimated results for different correlation structures are visible in Fig. 6. We choose estimating the bivariate models over the 3-dimensional model in order to obtain more precise estimates for the transition parameters in the TV-ETVC-GARCH(1,1) model and because valuable information for the correlation dynamics can be obtained by studying sub-models instead of higher-dimensional models. The estimation results have interesting interpretations. We observe that the short-run dynamic correlations obtained from the TV-EDCC-GARCH(1,1) model tend to fluctuate around the time-varying unconditional correlations estimated from the TV-ETVC-GARCH(1,1) model. The time-varying (un)conditional correlations show a decreasing trend over time, suggesting that the co-movements of the yield returns become weaker during the period of higher uncertainty. The long-term time-varying correlations show a downward trend movement descending to a level lower than that of before the global and sovereign debt crises and turn out to become smaller than “normal” market co-movements proxied by the constant conditional correlations.

8. Conclusions

We propose an additive structure for the extended vector GARCH process to investigate the dynamics of co-dependence volatility across financial markets. In this regard, we consider the general class of conditional correlation GARCH models where the volatility parameters are allowed to change smoothly over time by adding a deterministic time-dependent component to the variance equations. Within our approach, the timing of volatility regime changes is identified from a purely data-driven procedure.

For detecting changes in the volatility and co-volatility spillovers, we develop a Lagrange multiplier (LM) test for testing the hypothesis of parameter constancy where the rejection of the null hypothesis provides evidence for structural changes in the (co-)volatility processes. Crisis-contingent structural changes in volatility interactions can be interpreted as cross-market contagion, thereby the new test can be regarded as a volatility-based test of contagion. Simulation experiments show that the robust univariate LM-type test is reasonably well-behaved in finite samples.

Our modelling technique is applied to the study of the Greek, Irish, and Portuguese sovereign bond markets during the global financial crisis and subsequent sovereign debt crisis. We find the new statistical test to be a useful tool in model specification for the extended vector GARCH model. Results indicate a strong rejection of the parameter constancy hypothesis suggesting structural changes in the standard GARCH coefficients and co-volatility processes during the observation period. Once the model accounts for time-variation in the volatility interactions, the fit of the model substantially improves, and volatility persistence is remarkably decreased. The computational burden of the higher-dimensional model is alleviated by estimating the conditional variances equation by equation in the first step and the correlation matrix in the second step. Our estimation results further

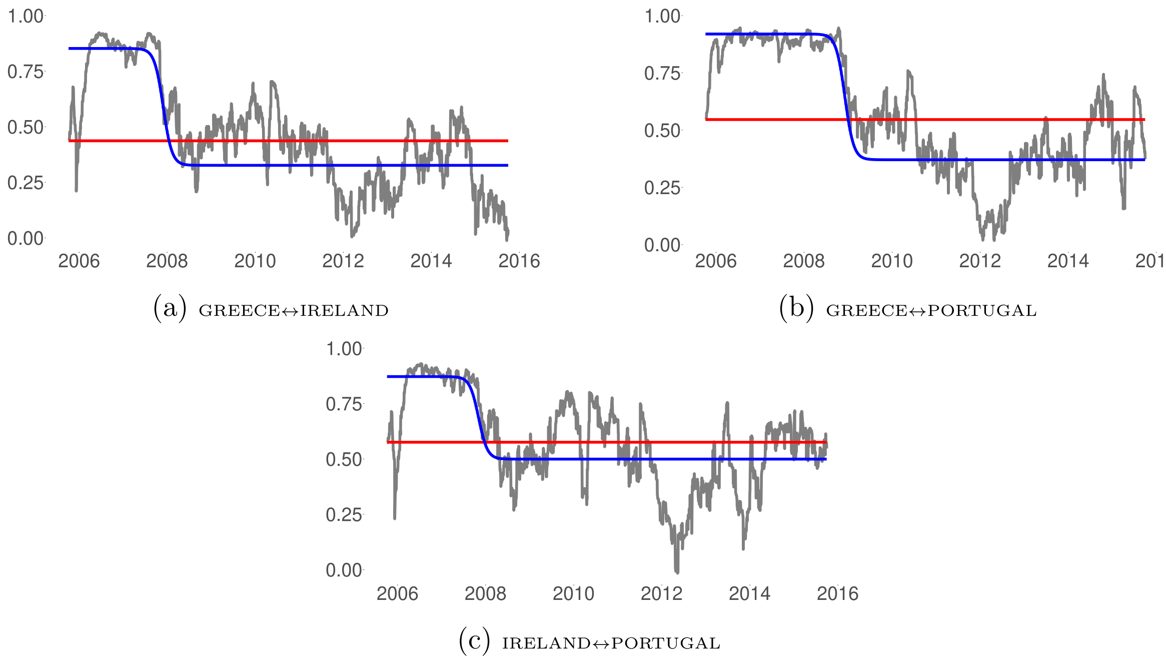


Fig. 6. Estimated bivariate dynamic conditional correlations (grey line) from the TV-EDCC-GARCH(1,1) model, time-varying unconditional correlations (blue line) from the TV-ETVC-GARCH(1,1) model and constant conditional correlations (red line) from the TV-ECC-GARCH(1,1) model.

suggest that volatility interactions appear stronger during the most acute phase of the European sovereign debt crisis. Volatility-based contagion is also identified bidirectionally for the pair Greece–Portugal and Ireland–Portugal, and unidirectionally from Greece to Ireland.

Although the tests for structural changes in the dynamics of volatility (or co-volatility) appear fairly robust against time-varying conditional correlations, it would be of interest to extend them to the extended conditional correlation GARCH model with time-varying correlations. Another interesting practical question would be to investigate how volatility responds to negative or positive shocks within this framework. A possible extension to the model would be to include the so-called leverage effect as in Francq and Zakoïan (2012). Given the weak response of conditional volatilities to the sign of shocks found empirically for bond returns (Cappiello et al., 2006), the effects of asymmetric volatility shocks were not considered in this work. The above issues are left for future research.

CRediT authorship contribution statement

Susana Campos-Martins: Conceptualization, Software, Validation, Formal analysis, Investigation, Writing – original draft, Funding acquisition. **Cristina Amado:** Conceptualization, Validation, Formal analysis, Investigation, Writing – original draft, Project administration, Funding acquisition.

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Appendix. The score and the hessian of the log-likelihood function

Under the assumption of normality, $\boldsymbol{\varepsilon}_t|\mathcal{F}_{t-1} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_t)$, the conditional log-likelihood function for observation t is defined as

$$\begin{aligned}\ell_t(\boldsymbol{\theta}) &= -(m/2)\ln(2\pi) - (1/2)\ln|\boldsymbol{\Sigma}_t| - (1/2)\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\varepsilon}_t \\ &= -(m/2)\ln(2\pi) - \ln|\mathbf{D}_t| - (1/2)\ln|\mathbf{P}_t| - (1/2)\mathbf{z}_t'\mathbf{P}_t^{-1}\mathbf{z}_t \end{aligned} \quad (48)$$

In order to define the first and second partial derivatives of (48), denote the score vector for observation t as $\mathbf{s}_t(\boldsymbol{\theta}) = \partial\ell_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ and let

$$\mathbf{s}(\boldsymbol{\theta}) = (1/T)\sum_{t=1}^{T}\mathbf{s}_t(\boldsymbol{\theta}) = (1/T)\sum_{t=1}^{T}\left(\frac{\partial\ell_t(\boldsymbol{\theta})}{\partial\boldsymbol{\vartheta}'}, \frac{\partial\ell_t(\boldsymbol{\theta})}{\partial\boldsymbol{\rho}'}\right)' \quad (49)$$

be the average score. The score vector of (48) has the following form

$$\mathbf{s}_t(\boldsymbol{\theta}) = (\mathbf{s}_t(\boldsymbol{\vartheta})', \mathbf{s}_t(\boldsymbol{\rho})')' \quad (50)$$

where $\mathbf{s}_t(\boldsymbol{\vartheta}) = (\mathbf{s}_t(\boldsymbol{\vartheta}_1)', \dots, \mathbf{s}_t(\boldsymbol{\vartheta}_m)')'$ is partitioned into $\mathbf{s}_t(\boldsymbol{\vartheta}_i) = (\mathbf{s}_t(\boldsymbol{\phi}_i)', \mathbf{s}_t(\boldsymbol{\varphi}_i)')'$, $i = 1, \dots, m$. The notation $\mathbf{s}_t(\hat{\theta})$ defines the score evaluated at the maximum likelihood estimator $\hat{\theta}$.

The analytical expressions of the first partial derivatives of (48) with respect to θ are given in the following lemma.

Lemma 8.1. *The blocks of the ith element of the score vector (50) for observation t have the following representation*

$$\mathbf{s}_t(\boldsymbol{\vartheta}_i) = \frac{\partial \ell_{it}(\boldsymbol{\theta})}{\partial \boldsymbol{\vartheta}_i} = -(1/2)\frac{1}{\sigma_{it}^2}\frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\vartheta}_i}\left(1 - \mathbf{z}_t'\mathbf{e}_i\mathbf{e}_i'\mathbf{P}_t^{-1}\mathbf{z}_t\right), i = 1, \dots, m, \tag{51}$$

$$\mathbf{s}_t(\rho) = \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \rho} = -(1/2)\frac{\partial \mathrm{vec}(\mathbf{P}_t)'}{\partial \rho}\{\mathrm{vec}(\mathbf{P}_t^{-1}) - \mathbf{P}_t^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}_t^{-1}\} \tag{52}$$

where the vec(·) *operator stacks the columns of the matrix underneath one another,* $\mathbf{e}_i = (\mathbf{0}_{i-1}', 1, \mathbf{0}_{m-i}')'$ *and*

$$\frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\vartheta}_i} = \left(\frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\phi}_i'}, \frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\varphi}_i'}, \frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\tau}_i'}\right)' \tag{53}$$

with

$$\frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\vartheta}_i} = \boldsymbol{\omega}_{it} + \sum_{k=1}^{p_i}(\beta_{ik} + \beta_{ik}^* G_i(t/T))\frac{\partial \sigma_{i,t-k}^2}{\partial \boldsymbol{\vartheta}_i} \tag{54}$$

where $\boldsymbol{\omega}_{it} = (\boldsymbol{v}_{it}', \boldsymbol{v}_{it}'G_i(t/T), \mathbf{g}_{\tau_{it}}'(g_{it}/G_i(t/T)))'$, *with* $\boldsymbol{v}_{it} = (1, \varepsilon_{i,t-1}^2, \dots, \varepsilon_{i,t-q_i}^2, h_{i,t-1}, \dots, h_{i,t-p_i})'$ *and* $\mathbf{g}_{\tau_{it}} = (g_{\gamma_i t}, \mathbf{g}_{c_i t}')'$ *with* $\mathbf{g}_{c_i t} = (g_{c_{i1}t}, \dots, g_{c_{ik_i}t})'$, $i = 1, \dots, m$. *The blocks of* $\mathbf{g}_{\tau_{it}}$ *are*

$$g_{\gamma_i t} = \frac{\partial G_i(t/T)}{\partial \gamma_i} = G_i(t/T)(1 - G_i(t/T))\prod_{k=1}^{k_i}(t/T - c_{ik})$$

$$g_{c_{ik}t} = \frac{\partial G_i(t/T)}{\partial c_{ik}} = -\gamma_i G_i(t/T)(1 - G_i(t/T))\prod_{l=1, l\neq k}^{k_i-1}(t/T - c_{il}), \quad i = 1, \dots, m.$$

Proof. The expressions for the blocks in (51)–(52) are proven in Appendix 12 of Silvennoinen and Teräsvirta (2017). ■

The population information matrix equals

$$\mathcal{I}_T(\theta_0) = (1/T)\mathsf{E}\mathbf{s}(\theta_0)\mathbf{s}(\theta_0)' = \mathsf{E}\mathbf{s}_t(\theta_0)\mathbf{s}_t(\theta_0)' \tag{55}$$

where $\mathbf{s}_t(\theta_0)$ is the score evaluated at the true parameter vector θ_0. The negative of the expected Hessian matrix evaluated at θ_0 equals

$$\mathcal{J}_T(\theta_0) = -(1/T)\mathsf{E}[\mathcal{H}_T(\theta_0)] = -(1/T)\mathsf{E}\sum_{t=1}^{T}\frac{\partial^2 \ell_t(\theta_0)}{\partial\theta\partial\theta'}. \tag{56}$$

The Hessian for observation t of the log-likelihood function (48) has the partitioned form

$$\mathcal{H}_t(\theta) = \frac{\partial^2 \ell_t(\theta)}{\partial\theta\partial\theta'} = \begin{bmatrix}\mathcal{H}_{\vartheta\vartheta,t}(\theta) & \mathcal{H}_{\vartheta\rho,t}(\theta)\\ \mathcal{H}_{\rho\vartheta,t}(\theta) & \mathcal{H}_{\rho\rho,t}(\theta)\end{bmatrix} = \begin{bmatrix}\frac{\partial^2 \ell_t(\theta)}{\partial\vartheta\partial\vartheta'} & \frac{\partial^2 \ell_t(\theta)}{\partial\vartheta\partial\rho'}\\ \frac{\partial^2 \ell_t(\theta)}{\partial\rho\partial\vartheta'} & \frac{\partial^2 \ell_t(\theta)}{\partial\rho\partial\rho'}\end{bmatrix} \tag{57}$$

where the elements for the sub-blocks in (57) are given in the following lemma.

Lemma 8.2. *The sub-blocks of the second-order partial derivatives of the log-likelihood function (14) in Section 3.1 for observation t are given by*

$$\mathcal{H}_{\boldsymbol{\vartheta}_i\boldsymbol{\vartheta}_j,t}(\boldsymbol{\theta}) = \frac{\partial^2 \ell_{it}(\boldsymbol{\theta})}{\partial\boldsymbol{\vartheta}_i\partial\boldsymbol{\vartheta}_j'} = -(1/4)\frac{1}{\sigma_{it}^2\sigma_{jt}^2}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i}\frac{\partial\sigma_{jt}^2}{\partial\boldsymbol{\vartheta}_j'}\left(\mathbf{e}_i'\mathbf{P}_t^{-1}\mathbf{e}_j\mathbf{e}_j'\mathbf{z}_t\mathbf{z}_t'\mathbf{e}_i\right) \tag{58}$$

for $i \neq j$, *i,j=1, ...,m,*

$$\mathcal{H}_{\boldsymbol{\vartheta}_i\boldsymbol{\vartheta}_i,t}(\boldsymbol{\theta}) = \frac{\partial^2 \ell_{it}(\boldsymbol{\theta})}{\partial\boldsymbol{\vartheta}_i\partial\boldsymbol{\vartheta}_i'} = -(1/2)\frac{1}{\sigma_{it}^2}\left(\frac{1}{\sigma_{it}^2}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i'} - \frac{\partial^2\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i\partial\boldsymbol{\vartheta}_i'}\right)\left(\mathbf{e}_i'\mathbf{P}_t^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{e}_i - 1\right) - (1/4)\frac{1}{(\sigma_{it}^2)^2}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i'}\mathbf{e}_i'\mathbf{P}_t^{-1}\left(\mathbf{I} + \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{z}_t\mathbf{z}_t'\mathbf{e}_i \tag{59}$$

for $i = j$, *i=1, ...,m,*

$$\mathcal{H}_{\boldsymbol{\vartheta}_i\rho,t}(\boldsymbol{\theta}) = \frac{\partial^2 \ell_{it}(\boldsymbol{\theta})}{\partial\boldsymbol{\vartheta}_i\partial\rho} = -(1/2)\frac{1}{\sigma_{it}^2}\frac{\partial\sigma_{it}^2}{\partial\boldsymbol{\vartheta}_i}(\mathbf{e}_i\otimes\mathbf{e}_i)'(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}_t^{-1}\otimes\mathbf{P}_t^{-1})\frac{\partial\mathrm{vec}(\mathbf{P}_t)}{\partial\rho} \tag{60}$$

and

$$\mathcal{H}_{\rho\rho,t}(\boldsymbol{\theta}) = \frac{\partial^2 \ell_{it}(\boldsymbol{\theta})}{\partial\rho\partial\rho'} = -(1/2)\frac{\partial\mathrm{vec}(\mathbf{P}_t)}{\partial\rho'}\{\mathbf{P}_t^{-1}\otimes\mathbf{P}_t^{-1} - (\mathbf{P}_t^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}_t^{-1}\otimes\mathbf{P}_t^{-1} + \mathbf{P}_t^{-1}\otimes\mathbf{P}_t^{-1}\mathbf{P}_t^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}_t^{-1})\}\frac{\partial\mathrm{vec}(\mathbf{P}_t)}{\partial\rho} \tag{61}$$

where ⊗ denotes the Kronecker product.

***Proof*.** See Appendix 12 of Silvennoinen and Teräsvirta (2017). The analytical expressions for each element of $\partial^2\sigma_{it}^2/\partial\boldsymbol{\vartheta}_i\partial\boldsymbol{\vartheta}_i'$ can be obtained by straightforward calculation. ■

The expressions for $\partial\mathrm{vec}(\mathbf{P}_t)'/\partial\rho$ depend on the form of the correlation structure. For further details, we refer to Bollerslev (1990) for constant conditional correlations, Engle (2002) for dynamic conditional correlations, and Silvennoinen and Teräsvirta (2015, 2017) when the (un)conditional correlations are assumed to change smoothly over time.

Data availability

Data will be made available on request.

References

Baillie, R. T., & Bollerslev, T. (1990). Intra-day and inter-market volatility in foreign exchange rates. *Review of Economic Studies*, *58*, 565–585.

Bauwens, L., Laurent, S., & Rombouts, J. V. K. (2006). Multivariate GARCH models: A survey. *Journal of Applied Econometrics*, *21*, 79–109.

Bollerslev, T. (1990). Modelling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model. *Review of Economics and Statistics*, *72*, 498–505.

Caporin, M., & Malik, F. (2020). Do structural breaks in volatility cause spurious volatility transmission? *Journal of Empirical Finance*, *55*, 60–82.

Cappiello, L., Engle, R. F., & Sheppard, K. (2006). Asymmetric dynamics in the correlations of global equity and bond returns. *Journal of Financial Econometrics*, *4*, 537–572.

Chiang, M. H., & Wang, L. M. (2011). Volatility contagion: A range-based volatility approach. *Journal of Econometrics*, *165*, 175–189.

Cifarelli, G., & Paladino, G. (2005). Volatility linkages across three major equity markets: A financial arbitrage approach. *Journal of International Money and Finance*, *24*, 413–439.

Conrad, C., & Karanasos, M. (2010). Negative volatility spillovers in the unrestricted ECCC-GARCH model. *Econometric Theory*, *26*, 838–862.

Dahlhaus, R., & Rao, S. S. (2006). Statistical inference for time-varying ARCH processes. *The Annals of Statistics*, *34*, 1075–1114.

Diebold, F. X., & Yilmaz, K. (2009). Measuring financial asset return and volatility spillovers, with application to global equity markets. *The Economic Journal*, *119*, 158–171.

Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, *50*, 987–1007.

Engle, R. F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, *20*, 339–350.

Engle, R. F., Ito, T., & Lin, W. L. (1990). Meteor showers or heat waves? Heteroskedastic intra-daily volatility in the foreign exchange market. *Econometrica*, *58*, 525–542.

Engle, R. F., & Susmel, R. (1993). Common volatility in international equity markets. *Journal of Business & Economic Statistics*, *11*, 167–176.

Francq, C., & Zakoïan, J. M. (2009). Bartlett's formula for a general class of nonlinear processes. *Journal of Time Series Analysis*, *30*, 449–465.

Francq, C., & Zakoïan, J. M. (2012). QML estimation of a class of multivariate asymmetric GARCH models. *Econometric Theory*, *28*, 179–206.

Francq, C., & Zakoïan, J. M. (2016). Estimating multivariate volatility models equation by equation. *Journal of the Royal Statistical Society. Series B. Statistical Methodology*, *78*, 613–635.

Granger, C. W. J. (1993). Strategies for modelling nonlinear time-series relationships. *Economic Record*, *69*, 233–239.

Hafner, C. M., & Herwartz, H. (2006). A lagrange multiplier test for causality in variance. *Economics Letters*, *93*, 137–141.
Hamao, Y., Masulis, R. W., & Ng, V. (1990). Correlations in price changes and volatility across international stock markets. *The Review of Financial Studies*, *3*, 281–307.
He, C., & Teräsvirta, T. (2004). An extended constant conditional correlation GARCH model and its fourth-moment structure. *Econometric Theory*, *20*, 904–926.
Hong, Y. (2001). A test for volatility spillover with application to exchange rates. *Journal of Econometrics*, *103*, 183–224.
Jeantheau, T. (1998). Strong consistency of estimators for multivariate ARCH models. *Econometric Theory*, *14*, 70–86.
Jung, R., & Maderitsch, R. (2014). Structural breaks in volatility spillovers between international financial markets: Contagion or mere interdependence? *Journal of Banking & Finance*, *47*, 331–342.
Karanasos, M., Ali, F. M., Margaronis, Z., & Nath, R. (2018). Modelling time varying volatility spillovers and conditional correlations across commodity metal futures. *International Review of Financial Analysis*, *57*, 246–256.
Karanasos, M., Paraskevopoulos, A. G., Ali, F. M., Karoglou, M., & Yfanti, S. (2014). Modelling stock volatilities during financial crises: A time varying coefficient approach. *Journal of Empirical Finance*, *29*, 113–128.
King, M. A., & Wadhwani, S. (1990). Transmission of volatility between stock markets. *Review of Financial Studies*, *3*, 5–33.
Leung, H., Schiereck, D., & Schroeder, F. (2017). Volatility spillovers and determinants of contagion: Exchange rate and equity markets during crises. *Economic Modelling*, *61*, 169–180.
Lin, W. L., Engle, R. F., & Ito, T. (1994). Do bulls and bears move across borders? International transmission of stock returns and volatility. *The Review of Financial Studies*, *7*, 507–538.
Lin, C. F. J., & Teräsvirta, T. (1994). Testing the constancy of regression parameters against continuous structural change. *Journal of Econometrics*, *62*, 211–228.
Ling, S., & McAleer, M. (2003). Asymptotic theory for a vector ARMA-GARCH model. *Econometric Theory*, *19*, 280–310.
Liu, T., & Gong, X. (2020). Analyzing time-varying volatility spillovers between the crude oil markets using a new method. *Energy Economics*, *87*, Article 104711.
Luukkonen, R., Saikkonen, P., & Teräsvirta, T. (1988). Testing linearity against smooth transition autoregressive models. *Biometrika*, *75*, 491–499.
Malik, F. (2022). Volatility spillover among sector equity returns under structural breaks. *Review of Quantitative Finance and Accounting*, *58*, 1063–1080.
Nakatani, T., & Teräsvirta, T. (2009). Testing for volatility interactions in the constant conditional correlation GARCH model. *Econometrics Journal*, *12*, 147–163.
Pedersen, R. S. (2017). Inference and testing on the boundary in extended constant conditional correlation GARCH models. *Journal of Econometrics*, *196*, 23–36.
Silvennoinen, A., & Teräsvirta, T. (2005). Multivariate autoregressive conditional heteroskedasticity with smooth transitions in conditional correlations. SSE/EFI Working Paper Series in Economics and Finance No. 577.
Silvennoinen, A., & Teräsvirta, T. (2009). Multivariate GARCH models. In T. Mikosch, J. P. Kreiß, R. A. Davis, & T. G. Andersen (Eds.), *Handbook of financial time series* (pp. 201–229). Berlin, Heidelberg: Springer Berlin Heidelberg.
Silvennoinen, A., & Teräsvirta, T. (2015). Modeling conditional correlations of asset returns: A smooth transition approach. *Econometric Reviews*, *34*, 174–197.
Silvennoinen, A., & Teräsvirta, T. (2017). Consistency and asymptotic normality of maximum likelihood estimators of a multiplicative time-varying smooth transition correlation GARCH model. CREATES Research Paper 2017−28.
Tchahou, H. N., & Duchesne, P. (2013). On testing for causality in variance between two multivariate time series. *Journal of Statistical Computation and Simulation*, *83*, 2064–2092.
Teräsvirta, T. (1994). Specification, estimation, and evaluation of smooth transition autoregressive models. *Journal of the American Statistical Association*, *89*, 208–218.
Teräsvirta, T. (1998). Modeling economic relationships with smooth transition regressions. In A. Ullah, & D. Giles (Eds.), *Handbook of applied economic statistics*. New York: Dekker.
Teräsvirta, T., Tjøstheim, D., & Granger, C. W. J. (2010). *Modelling nonlinear economic time series*. Oxford: Oxford University Press.
Tse, Y. K., & Tsui, A. K. C. (2002). A multivariate generalized autoregressive conditional heteroscedasticity model with time-varying correlations. *Journal of Business & Economic Statistics*, *20*, 351–362.
Wooldridge, J. M. (1990). A unified approach to robust, regression-based specification tests. *Econometric Theory*, *6*, 17–43.
Wooldridge, J. M. (1991). On the application of robust, regression-based diagnostics to models of conditional means and conditional variances. *Journal of Econometrics*, *47*, 5–46.