Asset Price Distributions and Risk Aversion

What Can We Learn from Options Market?

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Abstract

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This dissertation infers descriptive statistical measures from the estimated risk-neutral probability density functions derived from short-term out-of-the money monthly S&P 500 option prices in respective pre-crisis and crisis periods. The generalized beta distribution of the second kind, the mixture of lognormal distribution and the lognormal polynomial distribution comprise the three parametric methods used to estimate such density functions. The three estimated risk-neutral probability density functions tend to be negatively skewed, leptokurtic and exhibit roughly equal distribution mean values.

The constant relative risk aversion coefficient is computed through the method used by Liu et al. (2007) for quarterly risk-neutral densities. The pre-crisis constant relative risk aversion value is approximately 2.672 with a MLN distribution and 2.666 with a GB2 distribution, compared to constant relative risk aversion of 2.507 and 2.477, respectively, during the crisis period. The real-world densities became less skewed, less kurtic and contain a higher first-moment value than the risk-neutral densities. Results are fairly consistent with available academic literature.

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1. Introduction

In accordance with Bahra (1996), Jackwerth (1999) and Bodarenko (2000), Risk-neutral probability density functions (RNDs) are used for multiple purposes, namely, to price complex derivatives, to conduct monetary policy, to ascertain probabilities implicit in market expectations of the occurrence and non-occurrence of relevant political and economic events, and to infer investor risk preferences at a given point in time. RNDs express the density functions of the underlying asset price of options at maturity implicit in market option prices given current and representative investor expectations.

In this document I estimate RNDs from the existing relationship of derivative securities' premiums, namely S&P 500 European index options, and corresponding strike prices. I extract RNDs from three different parametric methods: lognormal-polynomial[1] (LP), generalized beta of the second kind (GB2) and mixture of lognormals (MLN) distributions. Although, the simplest RND estimation method is proposed by Malz (1995), who merely assumes that the underlying asset follows a jump-diffusion process,[2] and the RND is assumed to be the second derivative of the option pricing function with respect to strike price.

According to Ross (1976) the estimation of RND is possible only if investors are risk neutral and if market completeness is ensured within the corresponding options market. Economic agents' risk neutrality implies that investors are indifferent towards risk and demand no additional compensation to allocate their wealth to a risky investment opportunity. Market completeness demonstrates that there are sufficient assets within options markets to guarantee a non-null payoff irrespective of future economic states.

I research the recent financial crisis's impact on RND extracted from option prices in a pre-crisis period (2003-2006), and during a crisis period (2007-2011), specifically, by comparing RNDs from both time intervals.

I decided to focus on the financial crisis that began in the United States in 2007 with the subprime crisis and continues to persist, representing, according to quite a few economists,[3] the worst economic downturn since the Great Depression in the 1930s.

However, in current-day economies representative investors exhibit intrinsic personal preferences towards risk and require compensation to assume investments opportunities with accompanying risk that depends upon the representative investor's relative risk aversion (RRA)

[1] Also referred to as Hermite polynomial approximation.
[2] The underlying asset price is assumed to follow a stochastic process equivalent to a Bernoulli distribution whose jump dimension is non-stochastic and within which only one jump can occur.
[3] Roubini (cited in Ferrara P. 2012)

coefficient. Moreover, several authors (see, for example, Li (2007) and Lochstoer (2009)) indicate that the RRA tends to be a countercyclical variable. Therefore, RNDs should be converted to real-world densities (RWDs) in order to account for investor risk preferences.

I also compute the associated representative investor's RRA and RWD of the S&P 500 in respective pre-crisis (2002-2006) and crisis (2007-2011), through the Liu et al. (2007) method, in order to determine the respective changes in investor expectations and attitude towards risk due to crisis.

Once proper RNDs are obtained, the empirical results indicate that parametric RNDs exhibit a similar distribution shape for effective strike prices, first- and second- moment distributions as in Jondeau and Rockinger (2000). During crisis periods, each parametric distribution family's RNDs tend to be more negatively skewed and further from being normally distributed than pre-crisis. During both time periods under analysis, each parametric distribution is likely to be leptokurtic like in Bahra (1997). MLN and LP distributions become less leptokurtic during crisis periods. The inverse is borne out by GB2 distribution.

Moreover, the constant RRA estimates also change due to crisis similarly as in Smith and Whitelaw (2009), and accordingly, the RWD become less skewed, kurtic and present a higher first-moment distribution than the corresponding RND family likewise in Liu et al. (2007).

The abovementioned empirical results show that the financial crisis changes investor expectations since the statistical features of RNDs and RWDs families modify during crisis.

The rest of this thesis is organised as follows: section 2 reviews the most important papers among available literature addressing relevant topics. Section 3 identifies the data used in this dissertation. Section 4 illustrates the methodologies used in this document. Sections 5, 6 and 7 explain the empirical results obtained from RND, RRA and RWD estimates. Section 8 presents the conclusion.

2. Literature Review

2.1 Risk-neutral densities

Breeden and Litzenberger (1978) follow Ross (1976) and define the mathematical expression of RND as in equation **(1)** where $c(x)$ is the call pricing function, (r) the risk-free rate, (T) the time remaining until option maturity, and (x) the strike price.

$$RND = e^{rT} \frac{\partial^2 C(x)}{\partial x^2}. \qquad \textbf{(1)}$$

According to Bahra (1997), the above expression requires a call pricing function that must be differentiable twice with respect to the exercise price, monotonic, continuous, and convex. The foregoing conditions must be satisfied in order for the function to correspond to a RND.

Additionally, five base assumptions must be established: investor risk neutrality, the estimated probabilities must be strictly positive, the RND integral must total one, the market must be perfect[4], and no arbitrage opportunities can exist within the market. Absence of arbitrage opportunities precludes investors from earning any additional income by undertaking zero net cost investments in the market.

Given that RNDs are continuous functions, estimated RNDs are somewhat biased due to the lack of continuous option prices from effective trading. RNDs can be derived from the parametric or non-parametric techniques partially explored in the subsequent paragraphs.

2.1.1 Parametric methods

Parametric methods are RND estimation techniques that assume that the probability density functions of option prices are defined by a specific function and/or may assume the underlying asset price at option maturity in order to follow a precise process, for example, Malz (1995) and Bates (1996) apply parametric techniques.

The choice of processes followed by the underlying asset price is slightly limited since only a small number of processes leads to a closed-form RND solution.

Parametric RND functions' defining parameters are estimated in order to assure minimization of the squared difference between the market price of the options and theoretical option price

[4] Market perfection requires inexistence of trading costs, taxes, transaction costs and investors can borrow and lend at the risk-free rate.

defined by the applied model. Once parameters are estimated, the function can be utilized to price options at any strike price. In parametric RNDs functions, a higher number of parameters corresponds to greater flexibility, not requiring a significant volume of data for reliable estimations.

Jackwerth (1999) divides parametric methods into “generalized distributions methods”, “expansion methods” and “mixture methods.” I also include the implied volatility function method (IVF) in this section since it includes parametric and non-parametric features.

The first category is mainly composed of distributions defined by at least three parameters whose limiting cases tend to be standard distributions, e.g. lognormal or gamma distributions, once estimated parameters assume specific values. The second division of parametric techniques is represented by standard normal distributions associated with other functions to better capture data characteristics. The third parametric method is a combined densities method in which RND is produced by a weighted combination of two or more independent density probability functions. The fourth category fits a function to the market implied volatility embodied in market option prices. The RND is estimated from the Black-Scholes (1973) pricing function (BS) by the Breeden and Litzenberger (1978) technique. It is worth mentioning that the BS model assumptions are irrelevant in this case since the model is used only as a conversion tool.

2.1.1.1 Generalized distributions methods

GB2 distribution and Inverse Burr[5] (IB) distribution are examples of generalized distributions. The Burr-3 is a density function defined by two parameters used to estimate the probability densities of positive variables characterized by closed-form solutions of the percentile functions of distribution density functions. IB distributions are applied by Dutta and Babbel (2002), proving that these distributions cause less pricing errors and reflect market expectations better than standard lognormal distributions. The GB2 is developed by Bookstaber and McDonald (1987) and is a probability density function defined by four parameters used to estimate the distribution of US stock returns. The authors point out that additional uncertainty is generated by the computation of additional parameters and that GB2 distributions may not be closed under multiplication. However, GB2 mirrors stock returns over short horizons (less than 25 days) fairly well compared to lognormal distributions. The same is not applicable over long time horizons.

[5] Or Burr-3.

2.1.1.2 Expansion methods

Expansion methods frequently exhibit negative densities, so restrictions to bound moments and cumulants of RND are required to ensure positive densities. LP[6] distributions, Edgeworth Expansion Series (ES) and Gram-Charlier Expansions (GC) serve as examples of the parametric method under analysis. Under the LP method, the RND is approximated by a polynomial function multiplied by the standard normal density function. The stated polynomial function is made up of Hermite polynomials that usually are restricted until the fourth order to capture the first four moments of the RND distribution. The ES technique is very similar to the LP approach but instead of relying on distribution moments, distribution cumulants are used instead. The GC probability density functions are formed by the multiplication of the normal density function with two Hermite polynomials incorporating the skewness and kurtosis of the RND.

The LP distribution is developed by Madan and Milne (1994) to price and hedge contingent claims. The authors apply the method to short-term Eurodollar future options to avoid econometric inconsistencies and provide evidence of LP distribution superiority regarding lognormal distribution in options pricing error minimization. The superior performance of LP pricing is due to its capability to capture the first, second, third and fourth moments of the RND distribution whereas the BS model only captures the first two distribution moments.

Coutant, Jondeau and Rockinger (2001) analyse RND of the European future options over the Paris Interbank Offer Rate by the LP estimation technique and infer the impact from French political events on estimated RND. The authors demonstrate that the first two moments of the LP estimates of RND tend to be stable, but the third and fourth moments are unbalanced, possibly indicating lack of consistency in the skewness and kurtosis[7] pricing. In fact, markets attribute a price to skewness and positive excess kurtosis. The researchers find evidence that pricing options through Hermite polynomials eliminates the effect of the volatility smile and that inclusion of American options does not alter the empirical result. The method works better for short-lived options than for long-lived options.

Under this methodology the authors further assume that the drift rate and the volatility term of the geometric Brownian motion are constant, a hypothesis that is often violated (e. g., volatility clustering). They state that the choice of the process followed by underlying asset price is irrelevant since investors are only concerned with maximizing gains at option maturity. Also, they acknowledge that the Hermite approximations should be performed on options with

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[6] Also referred to as Hermite polynomial approximation.
[7] Third- and fourth-order Hermite polynomials constitute a proxy for the price of skewness and kurtosis within the market.

different maturities instead of a single maturity and that the assumed price process followed by the underlying asset is very simple.

They prove that estimated RNDs track investor expectations towards future development of underlying asset price, and, in fact, anticipate official announcements capable of influencing the underlying asset price. In addition, they recognize that market participants tend to attach a significant weight to remote events that have not taken place in the recent past with a virtually non-existent likelihood of occurrence.

As a base assumption, the ES technique requires that the first cumulant equal the future index price. Negative RNDs are frequently encountered that can be corrected by applying additional restrictions developed by Jondeau and Rockinger (2001). ES distributions always contain positive excess kurtosis, and if cumulants of an order higher than four are used several problems emerge; higher instability of RND moments of an order higher than 2, more uncertainty due to additional estimates of model parameters, multicollinearity problems and a more extended negative distribution region. This method is not particularly suitable for application to data with real probability distribution very close to normal distribution.

2.1.1.3 Mixture methods

The mixture methods are likely to be very flexible despite requiring the estimation of several parameters.

The proportion associated with each function is positive since it represents the conditional probability of a future economic/political event implied by current investor expectations that together must add up to one. The mixture of lognormals probability density functions (MLN) constitutes an example of the previously mentioned method in which the RND is defined by two or more lognormal distributed functions whose future underlying asset price distribution is supposed to be represented by a weighted function of lognormal distributions. This methodology guarantees non-negative densities that sporadically present more than one mode, allowing the RND of the underlying asset price to assume several different shapes even though no assumption is made concerning the process followed by the underlying asset price prior to maturity.

Melick and Thomas (1997) apply the MLN methodology to American crude oil futures options during the First Gulf War crisis period. The authors develop the definition of upper bounds for American options, concluding that the market was anticipating a significant change in prices and that market sentiment was better captured by a mixture of three lognormal functions than by a single function.

Jondeau and Rockinger (2000) indicate that MLN distributions are the best RND estimation model to gauge the market sentiment for short-lived exchange rate options but for long-lived options jump-diffusion models, such as Malz (1995), are better than the reaming analysed distributions. Also, Anagnou et al. (2002) apply the MLN methodology and conclude that RNDs estimated in the study through parametric distributions do not accurately represent the distribution of the underlying asset price at option maturity. This situation is verified for currency options and index options, although RND inaccuracy increases in the second case.

2.1.1.4 Implied volatility functions

Several functions can be applied to fit implied volatility functions (IVFs). Shimko (1993) fits a quadratic polynomial function to the lowest and highest strike price for all traded S&P 100 options maturing in different periods. The RND is extracted by combining the second derivative of the BS model call pricing function with respect to the strike price with the lognormal distribution fitted to the non-traded prices, although this procedure generates "kinks" or non-differentiable points in the RND. Malz (1997) takes a similar approach to that of Shimko (1993) but arrives at the IVF directly from the options' delta. Campa, Chang and Reider (1997) and Brown and Toft (1999) use cubic splines in order to increase the smoothing of the IVF function to better represent the option smile/smirk. Splines are IVF or pricing functions formed by a combination of different polynomials. Consequently, several parameters are estimated, and the respective fitting, knots and flexibility of the functions increase. The main advantage of the spline is that it is differentiable twice, represents the second derivative of the call pricing function, or IVF, and obviously constitutes a function. Nonetheless, Andersen and Wagner (2002) obtain negative RNDs estimated through splines.

2.1.2 Non-parametric methods

In contrast with formerly present methods, non-parametric methods do not require the process followed by the underlying asset price to be known because prices are assumed to be path-independent and the shape of the distribution is allowed to vary. Non-parametric techniques are especially applicable to situations in which the process followed by the underlying asset is unknown and there is a significant amount of data available to compute RND.

Non-parametric RND estimates tend not to capture the real distribution tails, are likely to allow arbitrage opportunities, lose several degrees of freedom in each estimate, are highly data intensive, and the quality of the model can be asymptotically inferred due to the large sample size.

The first flexible discrete distribution (FDD) technique is based on Arrow-Debreu securities,[8] and the RND is extracted from the probabilities associated with this variety of securities. The second methodology, known as kernel regression (KR), involves RND corresponding to a parameter-free function that is fitted to the pricing of options available on the market. The third technique is known as maximum entropy (ME). Taylor (2005) indicates that this method extracts the RND from option market prices constrained to the hypothesis that the option market prices are equal to option prices derived through the BS model and by previously assuming a specific distribution function. The fourth technique is the positive convolution approximation (PCA), in which RND is estimated through the integral of the product of two different simple functions.

2.1.2.1 Flexible discrete distribution

FDD is implemented by Rubinstein (1994). The probabilities that form the RND are derived from a binomial tree with n states, n different future asset prices and n different payoffs. However, the sum of the probabilities must add up to one, and the summation of the product of each individual probability and the corresponding future asset price must equal the forward price of the underlying asset at option extraction date. A significant degree of flexibility is allowed for the distribution shape.

Rubinstein (1994) applies FDD to estimate the IVF. Neuhaus (1995) improves the FDD methodology by developing an equation to capture distribution tails through use of the underlying asset price's cumulative distribution.

The Bhara (1997) technique assumes that RND probabilities can be derived through the compounded value of a butterfly spread at option maturity since it is assumed to yield the underlying asset value's discrete probability, equal to the strike price at option maturity.

2.1.2.2 Kernel regression

Härdle (1990) and Wand and Jones (1995) use kernel regression (KR) and Aït-Sahalia and Lo (1998) adopt this approach to price interest rate derivatives. Only applicable to time-series data and capable of accommodating several dimensions, the KR statistical technique represents the non-parametric method more commonly used within literature. The resulting KR function is differentiable twice with respect to strike price.

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[8] Assets that pay £1 under any given economic conditions and zero otherwise.

However, estimated RNDs tend to differ from those estimated through parametric methods and often lead to negative densities. Aït-Sahalia and Duarte (2003) develop a constrained model in order to demonstrate positive densities.

2.1.2.3 Maximum entropy

The maximum entropy (ME) technique is introduced by Jackwerth and Rubinstein (1996) and Buchen and Kelly (1996). This technique involves "extracting an asset's probability distribution by maximizing all unknown information, subject to the constraint of being consistent with all 'known' information," according to Guo (2001, p. 821), meaning that the method provides an estimate with minimum bias given the imperfect information built into market pricing by maximizing the information present in securities prices.

Buchen and Kelly (1996) demonstrate that as long as option strike prices are not homogeneous, the ME serves as a good proxy for the implicit RND. Jackwerth and Rubinstein (1996) derive RND from S&P 500 index options by employing this technique. Stutzer (1996) applies this method to derive RND from historical asset returns.

2.1.2.4 Positive convolution approximation

Bodarenko (2003) develops the Positive Convolution Approximation (PCA) technique. To estimate the RND, the first function comprised of the product of integrals is the kernel regression and second may be any simple function, e.g., the lognormal density. PCA can be applied to small sample sizes, increases the smoothness of the RND despite discrete option prices, is not high complex in a mathematical sense or time consuming and produces arbitrage-free pricing.

2.2 Risk aversion

So far, the representative investor has been considered to be risk neutral. However, Anagnou et al. (2002) show that RNDs present biased estimates of the index value at option maturity. They adjust the RND estimated from index options to reflect the representative investor's risk aversion in two simple ways: by arbitrary assumption of specific RRA coefficients and by adjusting the RND mean to reflect the market risk premium. Even with simple risk adjustments, the authors conclude that risk-adjusted RNDs fail to be considered unbiased estimators.

Moreover, Bliss and Panigirtzoglou (2004) and Liu et al. (2007) find evidence of RNDs untrustworthiness since the authors verify the existence of non-null RRA coefficients.

Accordingly, RND must be adjusted for risk to better reflect market expectations. The representative investor utility function and RRA coefficient are necessary to transform the RND into a RWD. The former two constitute the risk aversion adjustment. Villa and Pérignon (2002, p. 497) define the coefficient of relative risk aversion (γ) as "the investor assessment of risk that determines the behaviour followed by the economic agent in risky situations."

Jackwerth (2000) defines the relationship between real-world density,[9] risk-neutral density and risk aversion as follows:

$$RND = RWD \times Risk\ Aversion\ Adjustment^{10}. \qquad \textbf{(2)}$$

All techniques to estimate the RRA require the following assumptions: the representative market is complete and frictionless; the representative investor invests all his or her wealth in the representative market (the only risky investment available on the market); investors are rational economic agents that require compensation for assuming risk and hold homogeneous expectations towards future economic development. Further assumptions may be required on an individual basis.

There are three primary alternatives to estimate the RRA from option prices: the minimization of the Berkowitz test method (MBT), the maximum likelihood estimation method (MLE) and the non-parametric estimation method (NPM).

The Berkowitz test is a statistical tool that determines the choice of the optimal RRA coefficient. The minimization of the Berkowitz test method computes the risk aversion adjustment assuming a specific utility function for the representative investor and definition of a range of possible values of constant absolute risk aversion coefficients (CARA) or constant relative risk aversion coefficients (CRRA). If the specified utility function is exponential, the RRA depends upon the index value upon option maturity and the absolute risk aversion is constant. If the utility function is a power function instead, the RRA is constant but the absolute risk aversion varies according to wealth. The corresponding RWD and Berkowitz test are computed for all assumed CARA and CRRA series.

The initial RRA estimation methodology is formed by a procedure involving three successive phases. The MBT method under discussion first requires estimation of the RND from a sample of market option prices with an equal time to maturity, and, second, the definition of an

[9] Or alternatively denominated Subjective Probability Density function. According to Jackwerth (2000) it "corresponds to investors expectations concerning the likelihood of the occurrence of any given future condition".
[10] Or Pricing Kernel. It corresponds to the marginal utility of the representative agent at expiration date.

admissible range of CRRA and CARA coefficients with corresponding transformation of RND into RWD, and, lastly, indication of the optimal CRRA and CARA levels by selection of the CRRA and CARA values that simultaneously maximize the forecasting capabilities of the RWD given the verified index levels at option maturity, and minimize the associated Berkowitz statistical test.

The second method MLE estimates the CRRA by maximizing the log-likelihood of the index level at options maturity given the initial set of RND parameters. This is only possible if the assumed representative investor has a power utility function and if RNDs are estimated by GB2, LP and MLN distributions. CRRA is assumed to be independent of wealth levels or, alternatively, index levels at option maturity.

The third method NPM is slightly different from the remaining ones, since, the RRA is obtained from the implied RRA function. Estimates of implied RRA functions are derived from independent estimates of RWD, RND and corresponding first derivatives. Commonly, the RND is estimated from option prices and the RWD is computed from historical index prices. The mentioned densities are used to infer the aggregate utility function and the implied RRA function features. The RRA function is a non-parametric function that depends upon the index value at option maturity and the RWD is assumed to be constant over time. However, RND can be estimated through parametric methods.

2.2.1 Minimization of the Berkowitz test method

Bliss and Panigirtzoglou (2004) apply the first method and utilize the IVF parametric method, namely a spline, to estimate RND, simultaneously assuming stationary parametric utility functions of the representative investor, i.e., a power utility function and an exponential utility function. They transform the derived RND into a RWD, and then evaluate the risk-adjusted utility function with the Berkowitz test given actual outcomes. The authors implicitly assume that investors' utility functions are constant over time, and the RND and the RWD are allowed to vary across time. This technique is quite data intensive and requires non-traditional statistical techniques to run statistical analysis. However, according to researchers the Berkowitz test tends to approach one and overstate the p-values associated with different RRA coefficients. They conclude that the both CARA and CRRA risk aversion coefficients are robust estimates that vary according to the level of volatility and option time to maturity but are roughly the same for the two varieties of utility functions. Also, they discover that the risk aversion adjustment is well behaved consistent and of adequate scale across the S&P 500 and FTSE 100 markets.

Alonso et al. (2006) apply the MBT method to the IBEX index estimating RND by implementing MLN distributions and assuming three different implied risk aversion functions, explicitly, power utility function, exponential utility function and a stochastic discount factor to reflect investor habit formation. Habit formation is represented by a consumption ratio as in Abel (1990). The risk aversion coefficient is considered to be time invariant in the first two cases and the optimal CARA and CRRA values are selected in order to minimize the mentioned test. The risk aversion coefficient is considered to be time variant in the third case. The overall conclusion is that there is no significant difference between RNDs and RWDs for short horizon options over the sub-period considered. Nonetheless, a risk adjustment should be applied to RNDs estimated from longer maturity options since RWDs represent faithfully real densities right tails. The excess risk premium demanded by option holders is not as significant as predicted in earlier literature.

Assuming a negative exponential utility function instead of a standard exponential utility function, the MBT is also implemented by Kostakis, Panigirtzoglou, and Skiadopoulos (2011). They find evidence that RWDs allow a better investment optimization than RNDs in the S&P 500.

2.2.2 Log-likelihood maximization

The Liu et al. (2007) develop closed-form transformations to convert RND into RWD for the GB2 and MLN parametric distributions. The authors follow Bliss and Panigirtzoglou (2004) and compute splines as non-parametric estimators of RND even though no closed form solution of a RWD is encountered. They prove that RWDs provide more accurate estimates of future underlying asset price than historical densities estimated from FTSE 100 time series prices. They also demonstrate that RWDs tend to exhibit lower negative skewness and kurtosis than RNDs when both are estimated through GB2 distribution. They conclude that the spline method is a poorer estimator of the data in use than the alternative GB2 and MLN parametric distributions.

Shackleton et al. (2010) apply the MLE method, but the likelihood maximization is performed out of sample with future index levels. They transform RND into RWD by establishing assumptions defining utility functions and by assumption of risk-premium functions. They focus their research on high-frequency data corresponding to S&P 500 European options and conclude that for daily estimates high-frequency historical densities are accurate estimators of future index prices but RWDs are better estimators for longer time horizons.

2.2.3 Non-parametric methods

The subsequent papers present alternative possibilities for estimation of representative investor implied risk aversion functions since under NPM the representative RRA is not a single constant but rather assumed to be a function of the underlying asset price at option maturity.

Jackwerth (2000) applies the Jackwerth and Rubinstein (1997) method, a parametric method that provides the RND function of derivative securities prices through a IVF in which the IVF is a weighted combination of the sum of the square of the implied volatility associated with strike prices and the sum of the square of the standardized implied volatility associated with sample strike prices. The Gaussian kernel density of indexed returns estimates the RWD. The author proves that the representative utility functions are well-behaved, positive functions, decreasing with wealth, and that RND and RWD are asymptotically lognormal distributed, prior to the 1987 S&P 500 crash. Implicit RRA and absolute risk aversion (ARA) functions decrease as wealth levels increase and are positive functions. After the crash the RWD remains approximately lognormal distributed, while the RND becomes leptokurtic and skewed to the left. By assuming that RWD does not change, they hypothesize that investors risk aversion changes. After the crash, they conclude that the representative utility function is inconsistent with economic theory in which the coefficient of absolute and relative risk aversion is negative for certain wealth levels, implying that investors are “risk loving”. Corresponding ARA and RRA functions are sensitive to investor’s wealth and increase with specific wealth levels.

Aït-Sahalia and Lo (2000) implicitly estimate the risk aversion coefficient for S&P 500 since they define the RND through the parametric IVF method, assume a logarithmic utility function and estimate the RWD as kernel regression of a time series of continuously compounded past index returns. To estimate the IVF, the authors use the Nadaraya-Watson kernel as an estimator and find evidence that the market representative investor is risk averse and that investors’ RRA increases with positive and negative extreme index levels. Therefore, representative investor’s risk preferences are not constant.

Pérignon and Villa (2002) geometrically estimate the coefficient of relative risk aversion for high frequency CAC index options, through a modified version of Arrow-Pratt implied RRA expression. To do so, the authors estimate RND through the payoff of a butterfly spread and through the IVF through use of a KR, corresponding to the approach of Aït-Sahalia and Lo (1998). The RWD is derived utilizing the Gaussian kernel of historical returns of the underlying asset’s previous performance over the index level at maturity. The authors argue that a proper measurement of investor risk aversion must be determined during an economic agent life cycle, over a specific period of time and across different levels of wealth. Consistent with concave utility functions, the achieved relative risk aversion functions tend to be positive and decreasing. Also, RWD is less negatively skewed than RND. The mentioned densities are likely to be smooth, constant and stable through the time period analysed.

Furthering work conducted in the previous papers, Constantinides, Jackwerth and Perrakis (2005) verify whether the no-arbitrage opportunity hypothesis and market completeness assumption hold true within the S&P 500 market by analyzing the risk aversion adjustment. They develop unrestricted models in which, for instance, the risk aversion adjustment is assumed to be dependent upon the price process followed by the state-dependent index value and conclude that market hypothesis violations, namely market completeness, perfection, and absence of arbitrage opportunities, are more frequent with out-of-the-money options than with in-the-money options and that the BS model prices call options more effectively before the 1987 crash than after.

3. Data

The options used in this dissertation are European Index Options on the S&P 500 traded on the Chicago Board Options Exchange (CBOE) with 15 trading days to maturity whose expiration date corresponds to the first Saturday after the third Friday of standard option expiration months. I have extracted option contracts from 108 consecutive months to estimate the RND from January 2003 to December 2011. In turn, I have divided the sample period into a pre-crisis interval (2003-2006) and a crisis time horizon (2007-2011) in order to verify the financial crisis's effects on underlying future asset price distributions. The three abovementioned density functions, LP, MLN and GB2 distributions, are estimated under both time horizons.

To estimate the CRRA coefficient I have focused on options expiring in March, June, September and December, as in Anagnou et al. (2002), since those are the more frequently traded options contracts. A total of 20 quarterly periods is used to estimate independent CRRA coefficients corresponding to a before crisis[11] and during crisis epoch.

Dividend yield[12], options prices and S&P 500 closing prices were obtained from the OptionMetrics database on the Wharton Research Data Services website. Risk-free rates were extracted from the Datastream database and correspond to the three-month discount Treasury yield.

[11]To estimate the CRRA coefficient the pre-crisis interval spans from March 2002 through December 2006 in order to establish an identical time frame for the respective pre-crisis and crisis period.
[12] The dividend yield is utilised for multiple goals, for instance, to computation of put-call parity and densities' moments.

Following the approach of Bhara (1997), I use out-of-the-money call and put options prices. Put prices are converted into call option prices through the put-call parity. Option prices are the simple average of the daily closing bid and ask prices as universally performed in academic literature.

4. Methodology

4.1 Risk-neutral densities

The GB2, the MLN and the LP distributions used in this study to extract RND implicit in options prices are all parametric. Such distributions were chosen due to their high level of flexibility and capacity to capture the tails of utilized option premiums density, as commonly practiced in similar studies. Likewise, in line with an assumption of a power utility function, the CRRA is estimated in closed form. Also, small samples are sufficient to extract the RND, which improves analytical tractability and robustness of the cited methods.

Computation of the descriptive statistics (skewness, kurtosis and standard deviation) associated with each distribution requires the implementation of the integral in equation **(3)**. The approximation indicated in **(3)** and a gap of 2 between strike prices is used to calculate the integral.

$$\int_a^b (x - e^{(r-d)T} S)^N f(x)dx \cong \frac{b-a}{n} \sum_{i=0}^{n} \left(x_i - e^{(r-d)T} S\right)^N f(x_i) \; ; N > 1, \quad \textbf{(3)}$$

where N corresponds to the number of the distribution moment being computed, d symbolises the dividend yield, while b represents the maximum strike price within the sample, a constitutes the lowest strike price in the sample, S corresponds to the index closing price at the option price extraction date, r depicts the three-month treasury yield rate and $f(x_i)$ equals the relative frequency associated with the strike price x_i. The same notation and corresponding significance are valid throughout the document.

The following paragraphs provide a brief description of the RNDs estimation methods implemented and functions applied to compute the CRRA coefficients and RWDs.

4.1.1 GB2 distribution:

As Anagnou et al. (2002) explain, the base assumption of the GB2 method is that the shape of the probability density function derived from the option prices belongs to the generalized distribution family.

The GB2 distribution is defined by four different positive parameters: a, b, p and q, which define its shape and mean. Likewise, a defines the kurtosis of the distribution, becoming more peaked with higher a values. Next, p and q are the parameters that define the skewness of the distribution. According to Bookstaber and McDonald (1987), denomination b is the scale parameter, meaning that it defines distribution dispersion, where higher b values correspond to greater dispersion of the price probability function. In short, a, p and q parameters representing GB2 distribution have no practical meaning, but assure positive densities and are capable of capturing positive and negative skewness. The mathematical definition of the RND is expressed as:

$$GB2(x; a, b, p, q) = \frac{|a|x^{ap-1}}{b^{ap}B(p,q)\left[1+(\frac{x}{b})^{a}\right]^{p+q}}, \quad \textbf{(4)}$$

B is defined as:

$$B(p,q) = \Gamma(p) \times \Gamma(q) \div \Gamma(p+q), \quad \textbf{(5)}$$

in which Γ corresponds to the gamma function.

GB2 distribution moments do not exist when $n \geq aq$ and its kurtosis is infinite as cited by McDonald and Bookstaber (1987). Limiting cases occur when a tends to zero and when q tends to infinity. The GB2 distribution tends to a lognormal distribution and to a generalized gamma distribution respectively.

The four cited parameters are estimated in such a way that the total sum of the squared errors between the option prices verified on the market and the option prices obtained through the application of the GB2 option pricing formula are minimized. The objective function to minimize, denoted $G(x)$, is as follows:

$$G(x) = Min_{a,b,p,q} \sum_{i=1}^{n} \left[C(x_{i\,,}t) - \widehat{C_i}\right]^2, \quad \textbf{(6)}$$

in which $(\hat{C}_i)$ corresponds to the estimate of the call price through the implemented method and $C(x_{i\,,}t)$ stands for the market price of the call option.

The RWD is defined in **(7)** for this parametric method.

$$\tilde{f}_{S_T}(x) = GB2\left(x; a, b, p + \frac{\gamma}{a}, q - \frac{\gamma}{a}\right) = \frac{a x^{a\left(p+\frac{\gamma}{a}\right)-1}}{b^{a\left(p+\frac{\gamma}{a}\right)} B\left(p+\frac{\gamma}{a}, q-\frac{\gamma}{a}\right)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{\left(p+\frac{\gamma}{a}\right)+\left(q-\frac{\gamma}{a}\right)}}, \quad \textbf{(7)}$$

where $\tilde{f}_{S_T}$ corresponds to the RWD function.

4.1.2 Lognormal-polynomial distribution:

This technique is introduced by Madan and Milne (1994) and is based upon the assumption that the S&P 500 index price is lognormally distributed therefore index standardized returns (z) are normally distributed. According to Taylor (2005), this parametric method requires more mathematical computations than its counterparts.

The standardization of index returns requires finite variance ($\sigma^2 T$) with a specific mean ($\mu T - 0.5\sigma^2 T$) for the variation of forward prices. The equation representing the standardization is given by equation **(8)**.

$$z = \frac{\log\left(\frac{F_T}{F}\right) - (\mu T - 0.5\sigma^2 T)}{\sigma\sqrt{T}}, \quad \textbf{(8)}$$

where F_T corresponds to the forward value of the index at the option expiration date and F corresponds to the forward price of the index three calendar weeks before. It is also necessary to assume that $S_T = F_T$ in order to be able to estimate RNDs.

Under the LP method, the density function to price a contingent claim is a lognormal density distribution multiplied by a logarithmic function. In turn, equation **(9)** defines the stated distribution.

$$f_{S_T}(x) = \frac{1}{x\sigma\sqrt{T}} f_Z(z). \quad \textbf{(9)}$$

If the above holds, then the probability density function of z is expressed by **(10)** and b_j are the parameters to be estimated.

$$f_z(z) = \phi(z) \sum_{j=0}^{\infty} b_j H_j(z), \quad \textbf{(10)}$$

in which ϕ corresponds to the Gaussian distribution and H_j are equivalent orthogonal Hermite polynomials.

$$\int_{-\infty}^{\infty} H_i(z)H_j(z)\phi\,(z)dz = \begin{cases} 1\; if\; i = j \\ 0\; if\; i \neq j \end{cases}. \qquad \textbf{(11)}$$

The expressions of the mentioned Hermite-Polynomials initial fourth orders are defined below as:

$$H_0(z) = 1\; H_1(z) = z\; H_2(z) = \frac{(z^2-1)}{\sqrt{2!}}\; H_3(z) = \frac{(z^3-3z)}{\sqrt{3!}}\; H_4(z) = \frac{(z^4-6z^2+3)}{\sqrt{4!}}. \qquad \textbf{(12)}$$

In the special case developed by Madan and Milne (1994), where $j \leq 3$ and $J = 4$, the RND becomes:

$$LP(z) = \left[b_0 - \frac{b_2}{\sqrt{2}} + \frac{3b_4}{\sqrt{24}} + \left(b_1 - 3\frac{b_3}{\sqrt{6}}\right)z + \left(\frac{b_2}{\sqrt{2}} - \frac{6b_4}{\sqrt{24}}\right)z^2 + \frac{b_3}{\sqrt{6}}z^3 + \frac{b_4}{\sqrt{24}}z^4\right], \qquad \textbf{(13)}$$

in which z is defined as in **(8)**

Furthermore, b_0, b_1 and b_2 must be equal to 1, 0 and 0 respectively, the constrain function **(14)** must be satisfied and the Hermite-polynomials must be of order 4.

$$1 + \frac{\beta^3 b_3}{\sqrt{6}} + \frac{\beta^4 b_4}{\sqrt{24}} = exp^{-\mu T}. \qquad \textbf{(14)}$$

The Hermite polynomials price (b_3 and b_4) represent the market price for skewness and kurtosis of the continuous probability distribution of options with a specific maturity. Where:

$$\beta = \sigma\sqrt{T}. \qquad \textbf{(15)}$$

The limiting case of this method is the Black-Scholes model (1973), where, the mean is equal to the risk-free rate, with skewness and excess kurtosis assumed to be zero, i.e., b_3 and b_4 equal zero. Estimation of RND through LP method becomes more mathematically complex as the number of basis elements increases. Therefore, those items are usually constrained, as I indicated above. The method permits the inference of model quality to be made by econometric tests, often leading to negative RNDs and instable third and fourth moments. The distribution parameters are computed in a manner similar to that of the GB2 method.

$$G(x) = Min_{\mu,\sigma,b_3,b_4} \quad \sum_{i=1}^{n}\left[C(x_{i\,,}t) - \widehat{C_\iota}\right]^2. \qquad \textbf{(16)}$$

As advanced by Joundeau and Rockinger (2001), additional algorithms must be introduced when estimated RNDs are negative, i.e., the excess kurtosis of the variable z distribution belongs to the closed interval from 0 to 4 and the distribution skewness is between -1.0493 and +1.0493.

The RWD is defined as in Countant (1999):

$$\tilde{f}_{st}(x) = \frac{f_z(z)\left(\frac{x^{\gamma-1}}{\sigma\sqrt{T}}\right)}{S^{\gamma} e^{\left((\mu-0.5\sigma^2)T\gamma+T\gamma^2 0.5\sigma^2\right)}\left[1+\frac{b_3}{\sqrt{6}}\left(\sigma\sqrt{T}\gamma\right)^3+\frac{b_4}{\sqrt{24}}\left(\sigma\sqrt{T}\gamma\right)^4\right]}. \quad \textbf{(17)}$$

4.1.3 Mixtures of lognormal distributions:

In this methodology the extracted RND is assumed to be a linear combination of distinct lognormal density functions. Consequently, the respective call option prices represent a weighted average of the BS price associated with both distributions. Since I consider fifteen trading days to maturity options, I learned that the RND of the index future price could be well forecasted by only two distributions given that the conditional probabilities of an event to happen (p) are very small with two lognormal distributions over the first subsample. For overall RNDs, p is always below 0.44.

The RND is then defined as:

$$MLN(x|S_1, \sigma_1, S_2, \sigma_2, p) = p\psi(x|S_1, \sigma_1, T) + (1-p)\psi(x|S_2, \sigma_2, T), \quad \textbf{(18)}$$

in which σ_i correspond to the annual volatility of distribution i, S_i corresponds to the initial value of the index associated with distribution i and ψ to the lognormal density. All parameters are individually estimated to each distribution in which $S_0 = pS_1 + (1-p)S_2$ and $p \in [0,1]$. S_0 corresponds to the index level at option extraction date.

This additional constraint reduces the number of free positive parameters from five to four. The parameters to be estimated are $S_{1,}\sigma_1, \sigma_2$ and p and are determined in accordance with the previous methods.

$$G(x) = Min_{S_1,\sigma_1,S_2\sigma_2,p} \sum_{i=1}^{n}\left[C\left(x_{i\,,}t\right) - \widehat{C}_i(x_i\,)\right]^2. \quad \textbf{(19)}$$

It is worth mentioning that a special case of this distribution is the lognormal distribution in which the conditional probability equals zero.

The RWD is defined in this case as:

$$\tilde{f}_{s_t}(x) = MLN(x|S_1^*, \sigma_1, S_2^*, \sigma_2, p^*) = p^*\psi(x|S_1^*, \sigma_1, T) + (1-p^*)\psi(x|S_2^*, \sigma_2, T), \quad \textbf{(20)}$$

where $\tilde{f}_{s_t}(x)$ corresponds to the RWD, $S_i^* = S_i exp\ (\gamma T \sigma_i^2)$, $i = 1{,}2$ and

$\frac{1}{p^*} = 1 + \frac{1-p}{p}\left(\frac{S_2}{S_1}\right)^{\gamma} exp\ (0.5(\gamma^2 - \gamma)(\sigma_2^2 - \sigma_1^2)T)$.

4.2 Risk aversion estimation

The technique applied to estimate the CRRA is the MLE method. The objective function that Liu et al. (2007) provide is defined by equation **(21)**:

$$log\left(L\left(S_{T,1}, S_{T,2}, \dots, S_{T,n}\middle|\theta^*\right)\right) = \sum_{i=1}^{n} log\ (\tilde{f}_i(S_{T,i}\ |\widehat{\theta_i}, \theta^*), \quad \textbf{(21)}$$

in which $\tilde{f}_i$ denotes the RWD associated with day i and, L corresponds to the likelihood function and θ^* represents the risk adjusted RND parameters. In turn, $\widehat{\theta_i}$ correspond to the vector of estimated parameters that define the RND, which vary across time i and the parametric distribution used. Applying this method, the CRRA is estimated by log-likelihood maximization of the index value at n different option maturity dates ($S_{T,i}$) and for n different expiration dates given the parameters of the RND densities associated with contracts maturing at each particular date (t_i^*).

Therefore, no overlapping data is used with n different RNDs extracted at 15 trading days to option contract maturity. In addition, the restriction **(22)** must be satisfied, so each periodic option expiration date is the same or subsequent to the corresponding period option extraction date and on or prior than the following period extraction date.

$$t_i \leq t_i^* \leq t_{i+1}, \quad \textbf{(22)}$$

where t_i stands for the option extraction date.

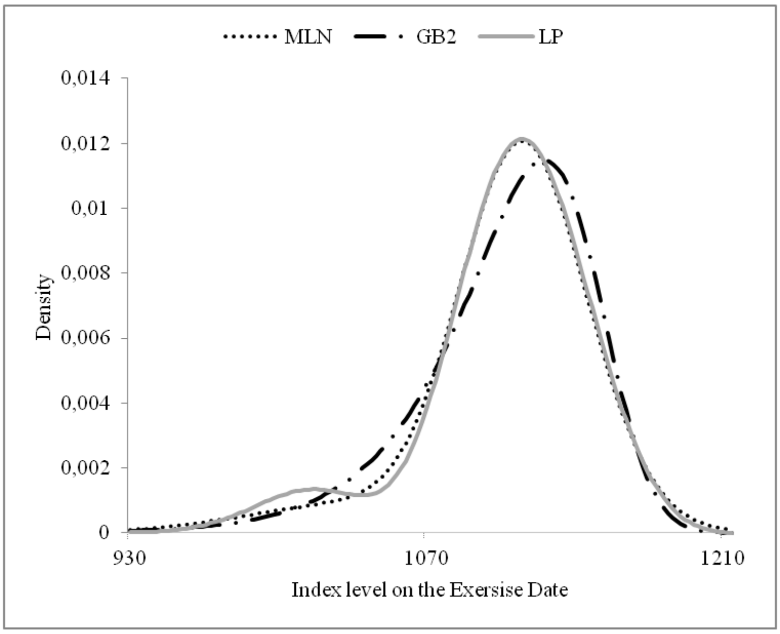
According to Liu et al. (2007), this technique is not inferior to the minimization of the Berkowitz test method in terms of quality and is also a very consistent and easily implemented method with good analytical tractability, while being relatively quick to run the necessary computations, as stated in the abovementioned Liu et al. (2007) method computation.

The CRRA coefficient is estimated individually for each parametric method used. To estimate the CRRA, it is still necessary to assume that the utility function of the representative investor is represented by a power utility function.

5. Pre-crisis risk-neutral densities

Within this time frame I have extracted data for 48 different and non-overlapping months. For each of them I have computed the three parametric distributions that persistently present a similar distribution shape within the traded strike prices, once proper distributions are encountered, as in Jondeau and Rockinger (2000). Figure 1 constitutes an example of analogous distribution shape.

Figure 1 – Illustration of three RNDs before crisis on March 26, 2004.



Likewise in Léon (2005), the LP probability density function presented negative probabilities in ten months. The Jondeau and Rockinger (2001) algorithms to restrict the kurtosis and skewness of the standardized variable z was capable of rendering seven of them fully positive. Frequently, the binding constraint was the negative bound of the skewness of z. In January 2003, August 2004 and December 2004, the LP probability density remains negative and therefore are excluded for inference purposes.

Table 1 – Pre-crisis monthly samples common composition.

Average number of option contracts used for monthly RNDs estimates before crisis	
Number of call options	23.67
Number of put options	34.94
Number of option contracts	58.60

According to Table 1, the average number of put options present in each monthly sample generally exceeds the number of call options by eleven units. The mean number of observations in each month is 59 option contracts.

The parameters of implemented distributions are estimated on a monthly basis by minimization of equations **(6)**, **(16)** and **(19)**, which correspond to the total sum of squared pricing errors of the difference between the market price of options and the estimated price of each parametric distribution variety henceforth Total G. The average values of equations **(6)**, **(16)** and **(19)** are shown in Table 2 under Total G for each parametric family. Higher Total G values correspond to lesser ability of the parametric method to correctly price options.

For quality inference purposes, the Total G should be adjusted to reflect the index levels and the number of options included in each RND monthly estimate. The G-adjusted[13] function considers both measures. Also, the number of minimizations performed through the use of different initial optimization parameters by using the Excel solver was not very prominent during the period and reasonable Total G values are encountered.

[13] $G - Adjusted = \sqrt{\left(\frac{Total\ G}{Number\ of\ options\ used\ to\ estimate\ RND}\right)} \times \left(\frac{1}{Index\ Value\ at\ Option\ ExtractionDate}\right)$

Table 2 – Summary statistics of average RND during a pre-crisis period.

	RND before Financial Crisis		
	GB2	MLN	LP
G-Adjusted	0.0002	0.0002	0.0002
Maximum	0.0006	0.0005	0.0004
Minimum	0.0001	0.0001	0.0001
Standard Deviation	0.0001	0.0001	0.0001
Median	0.0002	0.0001	0.0002
Total G	4.04	3.04	4.63
Maximum	29.83	27.99	29.41
Minimum	0.15	0.26	0.60
Standard Deviation	5.79	5.00	5.59
Median	2.91	1.65	2.79
Monthly Mean	1147.96	1148.85	1154.31
Maximum	1373.35	1372.25	1373.67
Minimum	838.26	838.30	838.30
Standard Deviation	131.11	131.58	128.43
Median	1168.57	1168.36	1169.83
Standard Deviation	38.65	38.98	39.39
Maximum	60.95	60.09	95.96
Minimum	30.20	28.99	30.54
Standard Deviation	7.71	7.48	11.11
Median	36.56	36.80	36.97
Skewness	-0.83	-1.00	-0.68
Maximum	0.21	0.17	-0.25
Minimum	-1.56	-1.88	-1.20
Standard Deviation	0.37	0.41	0.20
Median	-0.84	-1.02	-0.68
Kurtosis	5.43	6.40	4.55
Maximum	7.00	14.00	5.96
Minimum	3.33	2.85	1.64
Standard Deviation	0.74	2.36	0.65
Median	5.38	6.10	4.59
Jarque-Bera Test	25.48	57.16	12.41
Maximum	90.35	410.35	32.06
Minimum	0.85	1.12	0.74
Standard Deviation	16.48	75.23	6.68
Median	20.34	32.33	10.89

The summary statistics describe the characteristics of 45 LP distributions and 48 GB2 and MLN distributions derived from a pre-crisis period. The G-adjusted function corresponds to a fraction whose numerator corresponds to the square root of the total G function divided by the number of observations and the denominator to the index level at each extraction date. The monthly mean corresponds to the simple mean of all the monthly means included in the before crisis period. The skewness and kurtosis correspond to the average of the third moment and fourth moment of each distribution assortment.

In accordance with table 2, the average level of Total G is quite reasonable and similar across distributions, excluding the cases of negative LP distributions. GB2 distribution yielded more tackling problems given the higher level of programming errors, and a corresponding solution was rarely identified.

With respect to Total G, on average distributions that fit the sample option prices to poorer LP functions, while MLN fits data more effectively. LP distributions Total G increased considerably after applying Jondeau and Rockinger (2001) restrictions, as predicted in the literature.

MLN and GB2 distributions present a quite similar average first-moment distribution. The monthly mean is higher for the LP because fewer RNDs were included; otherwise, LP first-moment distribution would be closer to the remaining distributions first-moment.

Moreover, the preponderance of the RNDs typically exhibit excess kurtosis as Syrdal (2002) verifies. The more leptokurtic RNDs are on average the MLN, and LP distributions the least leptokurtic. Leptokurtic distributions are probability density functions with a more extended, higher peak than normal distributions.

When respects to skewness, all distributions are negatively skewed, excluding distributions estimated from options expiring in December 2004. The MLN present the longest left tails and LP the shortest left tails, indicating that the dispersion of outliers is higher on the left area of the distributions in the MLN case relative to LP.

According to the Jarque-Bera test, on average, the LP (MLN) distributions are closer to (further from) from the normal distribution, despite the fact that the vast majority of the distributions are not normally distributed at conventional significance levels. From the admissible RND set, twenty distributions can be considered normally distributed at a 5% significance level, while fourteen distributions can being considered normally distributed at a 10% significance level.

The highest level of kurtosis was reached in March 2006 by the MLN distribution and the lowest in August 2003 by the LP distribution. The main level of standard deviation corresponds to LP distribution in August 2003, while the tiniest is the MLN in December 2004. The skewness maximum occurred in December 2004 for the GB2, and the minimum is in June 2005 corresponding to the MLN. The peak value of the Jarque-Bera test occurs with the MLN in September 2006, and the lowest corresponds to the LP in May 2003.

Table 3 - Summary of statistics for average pre-crisis quarterly RND.

Pre-Financial Crisis RND Quarterly Data			
	GB2	MLN	LP
G-Adjusted	0.0002	0.0002	0.0002
Total G	5.30	3.16	5.20
Mean	1122.60	1122.76	1119.50
Standard Deviation	40.83	41.23	41.29
Skewness	-0.82	-1.07	-0.68
Kurtosis	5.67	6.77	4.86
Jarque-Bera Test	29.90	73.16	15.00

The summary statistics describe the average characteristics of 19 LP densities and 20 densities of GB2 and LP derived from a crisis period. The G-adjusted function corresponds to a fraction whose numerator corresponds to the square root of the total G function divided by the number of observations and the denominator to the index level at each extraction date. The mean corresponds to the simple mean. The skewness and Kurtosis correspond to the average of the third moment and fourth moment of each distribution.

In order to make the comparison with quarterly RWDs possible it is worth to analyse only quarterly RNDs over the same pre-crisis time interval.

When considering solely quarterly RNDs, within the pre-crisis period all descriptive statistics remained fairly similar and the ranking criteria remained virtually unchanged with respect to information shown in Table 3.

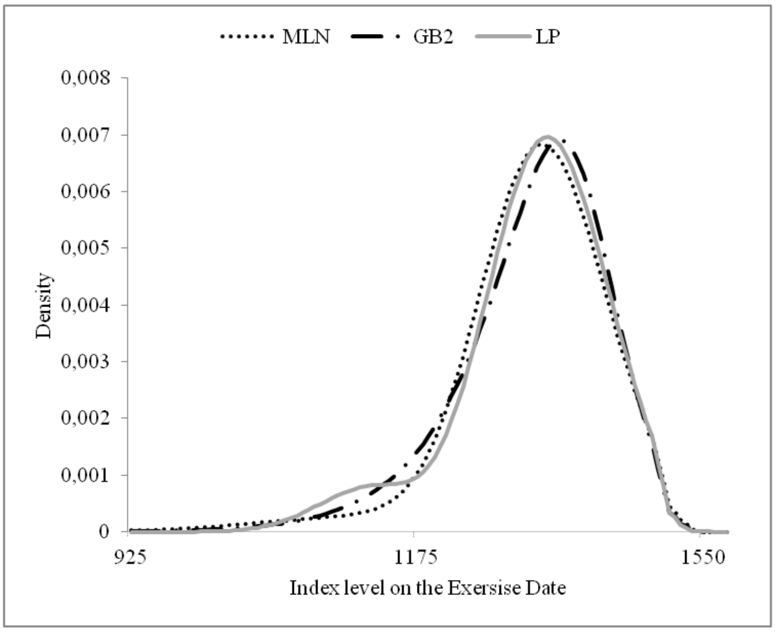
The only exceptions are the Total G and the G-adjusted in which the worst distributions to price options on average are the GB2 and the best are the MLN. Once again, the dissimilarity between the LP first-moment distribution and the other parametric distributions is only due to the exclusion of one LP negative RND.

Six negative RNDs are obtained under the LP method, and after applying restrictions to the z variable only one density function remained negative.

6. Risk-neutral densities during crisis periods

The crisis period contains samples extracted from sixty different and non-overlapping months. A total of 180 RNDs is computed. The shape of the computed RND is very similar within the traded strike prices provided estimated RNDs fulfil RND properties, as in McManus (1999). Figure 2 constitute an example of RNDs similarity.

Figure 2 – Three RNDs during crisis on October 28, 2011.



According to Table 4, the mean number of observations included in the monthly samples increased to 138, and the number of call and put options that constitute the average sample is more divergent than in the previous period.

Table 4 – Monthly samples typical composition during crisis period.

Average number of option contracts used for RNDs estimates during crisis	
Number of call options	52.52
Number of put options	85.58
Number of option contracts	138.10

Due to the amount of data used in each month, in order to reach the closest global minimum, the number of minimizations run through the excel solver increased substantially given that the quantity of error messages obtained increased and more local minimums were encountered. The average number of options used to estimate monthly RND during crisis exceeds the same average number of options utilised before crisis for a similar purpose.

The LP is the distribution that generated extra estimation problems and required more initial numbers of the optimization values, and, accordingly, supplementary Excel solver simulations, in order to achieve a proper result. The number of inappropriate RNDs is far more pronounced than in the previous case, i.e., 48 LP distributions presented negative probabilities, especially in the tails equally as in Countant (1999). The negative bound of the skewness of the variable z was the binding constraint yet again. After correction 39 LP distributions remained negative. As in the previous period, the Total G function value increases considerably post-correction, but the negative area of the LP distribution shrinks. The acknowledged previously, may indicate that the LP lacks flexibility to capture the recent financial crisis effects.

Table 5 – Summary statistics of average RND during financial crisis.

	RND during Financial Crisis		
	GB2	MLN	LP
G-Adjusted	0.0003	0.0003	0.0005
Maximum	0.0008	0.0008	0.0009
Minimum	0.0001	0.0001	0.0003
Standard Deviation	0.0001	0.0001	0.0002
Median	0.0003	0.0003	0.0005
Total G	23.41	21.80	60.99
Maximum	130.98	136.27	195.18
Minimum	1.91	2.63	13.52
Standard Deviation	22.66	21.10	58.71
Median	16.73	16.11	38.72
Mean	1210.02	1209.99	1258.12
Maximum	1534.77	1534.74	1534.09
Minimum	735.42	735.42	919.22
Standard Deviation	205.39	205.37	188.45
Median	1208.45	1208.46	1190.96
Standard Deviation	65.85	65.68	75.11
Maximum	128.89	128.26	306.87
Minimum	33.82	34.86	39.26
Standard Deviation	20.25	19.80	57.88
Median	60.71	60.79	60.16
Skewness	-1.18	-1.09	-0.75
Maximum	-0.64	-0.54	-0.12
Minimum	-1.75	-1.79	-1.46
Standard Deviation	0.29	0.33	0.51
Median	-1.26	-1.09	-0.81
Kurtosis	5.78	5.28	4.41
Maximum	7.26	9.99	5.67
Minimum	3.72	3.26	1.23
Standard Deviation	0.78	1.64	1.17
Median	5.94	5.03	4.53
Jarque-Bera Test	82.43	73.93	37.84
Maximum	210.36	360.25	83.10
Minimum	9.10	6.85	6.29
Standard Deviation	44.73	72.72	19.27
Median	68.35	50.56	35.70

The summary statistics describe features of 21 distributions LP and 60 GB2 and MLN distributions during a crisis period.

In Table 5 the MLN distribution presents the lowest value and the LP obtains the highest average Total G function. Therefore, MLN better fits the data characteristics than LP and GB2 distributions. MLN and GB2 distributions on average present a very close first- and second-moment distributions number, which is consistent with Jondeau and Rockinger (2000). The LP distribution does not present a similar first and second moment distribution, since a lower number of distributions is included.

Like in Constantinides et al. (2012), on average, each parametric RND family is leptokurtic. The most negatively skewed distributions and leptokurtic distributions tend to be the GB2 distributions, and the least negatively skewed and leptokurtic distributions are likely to be LP distributions, as shown in Table 5. GB2 distributions have the most pronounced number of outliers on the distribution's left region. In the crisis horizon, the entire distributions are negatively skewed, as in Figlewski (2008), and each parametric distribution family becomes more negatively skewed during the crisis compared to the pre-crisis period. Each family of parametric distributions encloses a more pronounced left tail during crisis than throughout the pre-crisis period.

Within this time interval, the estimated distributions fail to be considered normally distributed for all conventional significance levels. According to Table 5, the LP method is closer to normal distribution than its GB2 counterpart. However, it is worth mentioning that results associated with the LP distribution are slightly biased since only 21 distributions are considered, while sixty distributions are considered for the other two parametric methods.

The dip value of the Total G is reached in September 2007 by the GB2 distribution, and the peak by LP in June 2010. The lowest second-moment distribution value was achieved in March 2007 by the GB2 distribution, and peaked in April 2010 by the LP distribution. The smallest fourth-moment value occurred in April 2010 within the LP distribution, and the highest value occurred in May 2007 within the MLN. The skewness reaches its lowest point in March 11 by the MLN distribution, and peaks in August 2009 on the LP distribution. The minimum value of the Jarque-Bera test takes place in August 2007 according to the LP method, and the maximum is reached in March 2010 with the MLN.

Table 6 - Average summary statistics of quarterly RND during the crisis period.

RND during Financial Crisis Quarterly Data			
	GB2	MLN	LP
G-Adjusted	0.0003	0.0003	0.0005
Total G	25.55	20.93	64.16
Mean	1202.54	1202.51	1209.92
Standard Deviation	66.31	66.32	59.48
Skewness	-1.13	-1.10	-0.83
Kurtosis	5.74	5.48	4.80
Jarque-Bera Test	82.99	89.52	41.36

The summary statistics present the average descriptive statistics of 10 densities LP and 20 densities of GB2 and MLN derived from a crisis period. from 2007 until 2011. The G-adjusted function corresponds to a fraction whose numerator corresponds to the square root of the total G function divided by the number of observations and the denominator to the index level at each extraction date. The mean corresponds to the simple mean. The skewness and kurtosis correspond to the average of the third moment and fourth moment of each distribution.

As in the before the crisis epoch, through the analysis of table 6, the average summary statistics do not change considerably when only four months in each year belonging the crisis period are considered. Namely, all the descriptive statistics remain generally the same and the performance of each parametric distribution is kept on an average basis. The sole exception is the Jarque-Bera test in which the MLN distributions are more distant from being normally considered to replace the GB2 distributions.

The MLN and GB2 distribution moments are very similar, and, again, the dissimilarity of the LP distribution average distribution moments is due to the exclusion of some RNDs. The LP distribution is negative during thirteen periods, but application of the Jondeau and Rockinger (2001) correction renders three completely positive.

When only four months in each year are considered, on average, the skewness of each parametric distribution family is more negatively skewed during the crisis period compared to the pre-crisis period, indicating that investor expectations change during the crisis. The representative investor expects the occurrence of more outliers in the left region of the RND during the crisis period compared to the pre-crisis period. Additionally, investor expectations during the crisis period are more dispersed than the pre-crisis period due to higher relative standard deviations experienced in the crisis period. As such, economic agents during crisis periods are more uncertain in relation to the index level at options maturity than during the pre-crisis period.

All parametric distribution families in both intervals under the normality test tend to be far from being normal distributions, especially during the crisis period.

Further details of dispersion measures and the probability density functions are found in the appendix.

7. Risk aversion

One of the aims of my research is to compare and estimate the CRRA in periods both before and during a crisis. The subsequent table indicates the values obtained through application of the Liu et al. (2007) method and associated assumptions.

Table 7 – Coefficients of relative risk aversion estimates.

Coefficient of relative risk aversion		
	GB2	MLN
Before crisis	2.666	2.672
During crisis	2.477	2.507

Smith and Whitelaw (2009) state that the coefficient of RRA is a countercyclical variable. Therefore, during periods of economic expansion the RRA coefficient tends to decrease, while the RRA coefficient tends to increase during recessionary periods. Results from this study indicate that the RRA coefficient changes due to the financial crisis, albeit in the opposite direction.

However, the similarity and inverse sign variation of the CRRA is may be due merely to the pre-crisis and crisis period selection that can be biasing the results, since both intervals were defined in order to have an even number of periods in each sample instead of precisely capturing the beginning and end of the financial crisis and economic expansion. Therefore, during the financial crisis there are non-recession months considered that may unduly affect the results. Also, this slight disparity may be explained by small number of time intervals considered, whereas the RRA estimates from asset pricing models are applied through more successive time periods. Both factors corroborate a decline of CRRA during the crisis period.

Nonetheless, Liu et al. (2007) state that inclusion of the early 2000s recession reduces the RRA coefficients for the FTSE 100 to roughly half of the RRA coefficient estimates of Bliss and Panigirtzoglou (2004). My estimates of CRRA are consistent with Liu et al. (2007) since the inclusion of a recession in a sample composed of expansion and recession periods decreases the CRRA coefficients during the crisis period, as borne out by my research.

Moreover, Alonso et al. (2006) indicate that the CRRA should decline if the time to maturity of the option contracts decreases and encounter consistent results in the case of the habit formation model. Bliss and Panigirtzoglou (2004) find evidence of precisely the contrary, namely, for the same time period they estimate a CRRA for options with 3 weeks to maturity of 6.85 versus a 4 week to maturity CRRA of 4.08 for the S&P 500.

The mentioned on the previous paragraphs adds consistency to the values of RRA reached before and during crisis. The RRA coefficients are in both time intervals superior than the RRA presented by Liu et al. (2007) since the authors are utilizing 4-week to maturity options and I am working with 3-week to maturity options which is consistent with Alonso et al. (2006).

Furthermore, Smith and Whitelaw (2001) indicate a RRA coefficient of 3.33 during recessions and in a more recent paper, Smith and Whitelaw (2009), achieve during recessions a RRA coefficient of 3.1. The later estimate includes more periods of negative economic growth which may indicate that the RRA coefficient is downwards sensitive to the dimension and frequency of economic recessions during the time interval considered for RRA estimates. Consequently, the higher the frequency and economic contraction the lower will be the RRA. Therefore, I can conclude that, in fact, during economic expansions the RRA coefficient decreases since the estimated pre-crisis CRRA falls below the numbers that Smith and Whitelaw present in both papers. I am additionally able to deduce that due to the severity of the recent financial crisis and the lengthy period of economic recovery within the crisis period, the CRRA during the crisis should be lower than the RRA coefficient of Smith and Whitelaw, which is indeed borne out by the research.

Additionally, Liu et al. (2007) present similar CRRA coefficients estimated by applying the MLE method to GB2 and MLN distributions. The authors estimate a CRRA of 1.85 with a MLN distribution and a CRRA of 1.86, along with a GB2 distribution. Therefore, estimates of CRRA under the two different parametric distributions should be similar under the same time periods. In fact, my estimates of CRRA are very alike within both time periods, adding even more consistency to present CRRA estimates.

Finally, RRA values in Table 8 obtained in both periods are consistent with the vast majority of the papers presented, i.e., Normandin and St-Amour (1998), which affirm that RRA should be less 3, as occurring during both periods. Friend and Blume (1975) state that the RRA coefficient should be higher than two which is also consistent with the CRRA obtained in the present study to fulfil the requirement of being higher than two. Finally, the CRRA estimates for the pre-crisis and crisis periods are within the ranges that Guo and Whitelaw (2006) identify for the asset pricing models that they develop.

Table 8 – Relative risk aversion coefficients estimates extracted from the available literature.

Study	Market	Period	RRA Range
Friend and Blume (1975)	Com. Stock Index /S&P	1902-1971	> 2
Normandin and St-Amour (1998)	NYSE	1959-1992	< 3
Smith and Whitelaw (2001)	S&P500	1983-1998	3.33
Pérignon and Villa (2002)	CAC	1999	4.27
Guo and Whitelaw (2003)	S&P100	1983 – 1995	-.463 – 4.83
Bliss and Panigirtzoglou (2004)	FTSE100	1992-2001	3.95
Guo and Whitelaw (2006)	S&P100	1984-2001	2.485 – 5.916
Guo (2006)	CRPS	1952-2000	1.664 – 6.441
Liu et al. (2007)	FTSE100	1993-2003	1.85 and 1.86
Smith and Whitelaw (2009)	CRPS	1952-2005	3.1

Table 8 exhibits RRA coefficients determined by the most relevant papers within the literature, but it is worth mentioning that the highlighted RRA are derived both from asset pricing models and option prices. This table is partially replicated from Bliss and Panigirtzoglou (2004).

The CRRA estimated through LP distributions are significantly different from the CRRA of the previous methods, namely, .63 after crisis and -0.04 before crisis. I will not use these CRRA due to the reduced sample size after eliminating the negative densities and apparent unreliability of estimates.

7.1 Real-world densities

As mentioned above, several academic studies submit evidence that RNDs do not accurately represent investor expectations of the future underlying asset price for long maturity options. Since investments into the S&P 500 have not sure outcome investors have to make decisions under uncertainty and they demand a risk premium to allocate their wealth into a riskier investment.

The estimated CRRA constitute the price demanded by the representative investor for undertaking risky investments. In the following section I analyse the transformed distributions (RWDs) once the corresponding RNDs are corrected by the estimated CRRA. The representative utility function is a power function under the MLE. Additionally, comparison between RNDs and RWDs of each parametric family will be performed, in this section.

Table 9 - Summary of statistics for pre-crisis and during crisis RWDs.

	Real-World Densities			
	Pre-Financial Crisis		During Financial Crisis	
	GB2	MLN	GB2	MLN
G-Adjusted	0.0101	0.0026	0.0140	0.0062
Total G	11716.82	596.94	84991.89	9685.03
Mean	1147.87	1134.25	1286.12	1215.09
Standard Deviation	59.76	39.53	108.08	61.88
Skewness	0.77	-0.56	0.63	-0.54
Kurtosis	2.91	6.03	2.73	5.09
Jarque-Bera Test	10.47	44.05	26.33	50.42

The summary statistics describe the average characteristics of 20 densities of each family derived from a pre-crisis crisis period on the left-hand side and from a crisis period on the right-hand side. The G-adjusted function corresponds to a fraction whose numerator corresponds to the square root of the total G function divided by the number of observations, while the denominator refers to the index level at each extraction date. The mean corresponds to the simple mean. The skewness and Kurtosis refer to the average of the third moment and fourth moment of each distribution.

Within a pre-crisis period, both distributions shape change significantly. The GB2 distribution is the method that exhibits a higher average Total G, indicating that RWD estimated through MLN distributions capture the periodic sample characteristics after considering the CRRA better than GB2 distributions. The G-adjusted function is always below 0.02 for GB2 and below 0.01 for MLN. The distributions with highest standard deviation and expected value are the GB2. MLNs present the lowest previously mentioned dispersion and localization measures. The MLN correspond to more negatively skewed distributions with higher kurtosis. GB2 distributions are more positively skewed and less kurtic.

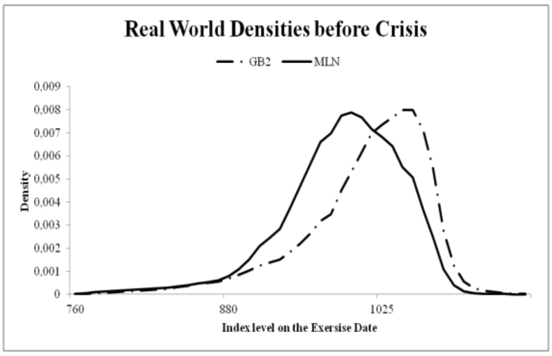
When referring to the crisis period, distribution shapes change even more noticeably than in the pre-crisis period. However, the comparison of the two distributions remains similar in terms of the statistical descriptive measures under evaluation. The GB2 distributions exhibit a higher number of pricing errors, indicating that RWD estimated through a GB2 distribution accommodates the observation prices less effectively than MLN distributions. The G-adjusted function is always below 0.05 for GB2 case and below .025 for MLN. GB2 tends to present a higher mean value and second-moment distribution than MLN. The latter corresponds to more negatively skewed distributions, while GB2 is associated with more positively skewed distributions. With respect to kurtosis, MLN distributions present a higher kurtosis value than GB2.

In terms of proximity to the normal distribution in the both time periods under consideration (the respective pre-crisis and crisis epochs), the average distributions are not deemed to be normal for all conventional levels of significance. However, the GB2 distribution is closer to normal than the MLN during the pre-crisis period.

During crisis a similar conclusion is reached, namely that on average GB2 and MLN distributions cannot be considered normal distributions for all the conventional significance levels. However, GB2 distributions are closer to normal distribution than MLN distributions. Nonetheless, RWDs estimated by GB2 distributions are closer to be approximately normal pre-crisis than during the crisis period since the associated Jarque-Bera test is lower in the pre-crisis period than during the crisis. The same pattern is verified by the MLN distributions, where on average pre-crisis MLN distributions are closer to normal than during the crisis period.

Specifically, pre-crisis there are seven distributions capable of being considered approximately normal at a 5% significance level and two distributions at a 10% while in the crisis epoch there are six and five distributions correspondingly.

Figure 3 and Figure 4 constitute examples of the RWDs obtained during the respective pre-crisis period and crisis periods.



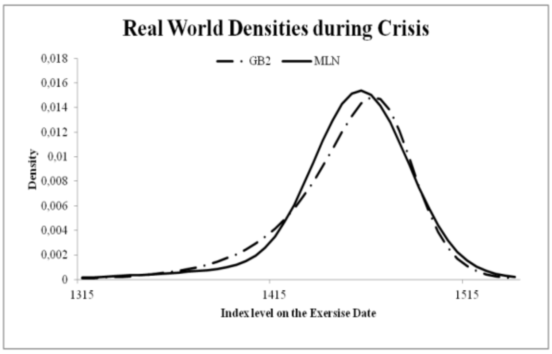


Figure 3 – Pre-crisis RWDs on November 29, 2002. Figure 4 – RWDs during crisis on February 23, 2007.

The graphs above depict examples of RWDs extracted from a pre-crisis and crisis periods, respectively. The dotted line corresponds to RWD for GB2 distributions and the solid line to RWD for MLN distributions.

In conclusion, it is appropriate to contrast the estimated RWD with the corresponding RND family, i.e., to analyze the differences between the RND estimated through GB2 (MLN) distribution and the RWD obtained through the GB2 (MLN) method, both during and prior to the crisis period. Tables 3, 6 and 9 are subsequently analyzed.

With regard to the pre-crisis period, my results for both distributions are consistent with the literature. Namely, the transformed densities on average exhibit less skewness[14], average Jarque-Bera statistics and higher average first-moment distribution than the RND estimates for the same method. On an individual basis, further literature consistency examples are verified, namely that RWD estimates through MLN distributions exhibit average standard deviation

[14] Throughout the RWD section I mean less negative skewness in the MLN case and less positive skewness in the GB2 case when stating less skewness.

which is smaller than the corresponding RND value. Moreover, RWD estimated by the GB2 distribution mean kurtosis is lower than the matching average RND distribution kurtosis.

Among the remaining empirical results during the pre-crisis period, RWDs estimated through MLN present less kurtosis than RNDs estimated through the same distribution, and RWDs estimated through the GB2 method present a higher average second-moment distribution value than the corresponding second-moment RNDs.

When comparing RWD to RND summary statistics are the same during both the pre-crisis and crisis periods. In other words, the skewness is reduced in both distributions, and the GB2 distribution becomes positively skewed. The kurtosis and the Jarque-Bera statistics decrease considerably. The content of the last two sentences is consistent with empirical results presented by Liu et al. (2007). Finally, the RWD standard deviation under GB2 distributions does not decrease, unlike MLN.

8. Conclusion

RNDs extracted from option prices reflect market expectations of the representative investor regarding future underlying asset price. Based on analysis of RND estimated from use of parametric methods, the GB2, MLN and LP distributions, all the implemented distributions lead to similar distribution shapes within the traded strike prices range and similar estimates of first- and second-moment RND, which is consistent with the cited literature both before and during crisis, when all three parametric distributions present positive densities. In the case of the LP distribution method, distributions during the crisis period frequently present negative density areas. Likewise, numerous academic papers verify the occurrence of negative densities.

RNDs estimated through different families of parametric distributions before the crisis period tend to be less negatively skewed than the corresponding RND family during a crisis period, indicating that investor expectations changed due to crisis given their expectation that more negative outliers would occur during compared to before the crisis. Also, when referring to RND normality, pre-crisis distributions are closer to normal distribution than during a crisis period. Prior to the latter, there are few normal distributions, while during a crisis period, approximately normal parametric distribution is considered to be absent.

CRRA estimates are fairly consistent with the literature and reasonably represent the price demanded by the representative investor for each additional unit of risk. CRRA changes following the financial crisis yield similar CRRA with GB2 and MLN distributions. The pre-crisis CRRA is approximately 2.672 with a MLN distribution and 2.666 with a GB2 distribution, whereas the CRRA is 2.507 and 2.477 during the crisis period for the respective MLN and GB2 figures.

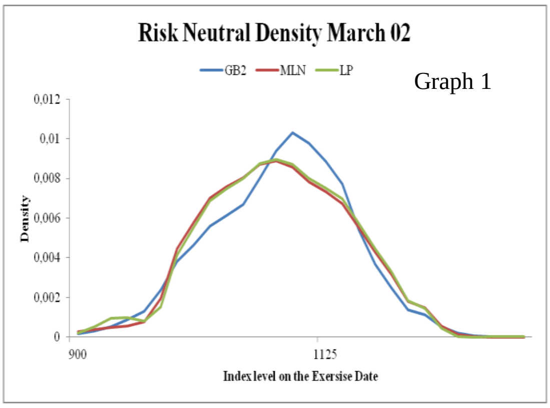
Consistent with available empirical literature, RWD transformed from a specific parametric family tend to be less skewed and present a lower Jarque-bera test value than the corresponding RND from the same parametric family. In fact, GB2 distributions during a pre-crisis and crisis period, respectively, present positive skewness in the aggregate. Also, adjusted densities families exhibit closer-to-normal distribution relative to the corresponding RND. The results obtained in this dissertation include evidence from existing literature in this field of research, such as Liu et al. (2007).

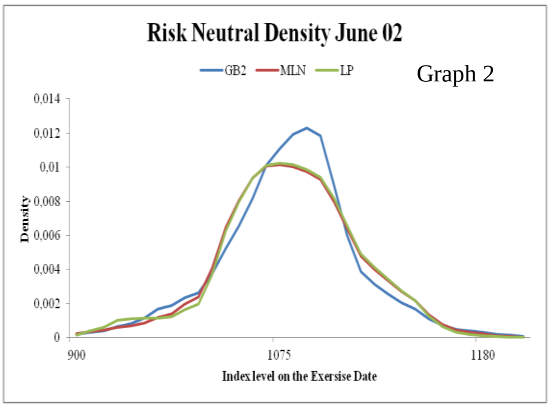
Furthermore, when comparing RWD prior to and during a crisis, respectively, it is important to highlight that the descriptive statistical measures do not change dramatically. However, transformed RNDs capture different market conditions since the distribution shape for the effectives strike prices changes slightly.

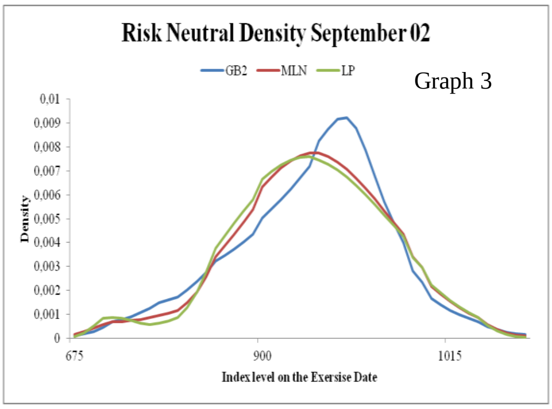
For purposes of future empirical researches, it would be appropriate to estimate the CRRA by using all monthly densities in a future study. Further research covering options three weeks prior to maturity would also be relevant in order to make CRRA comparisons possible.

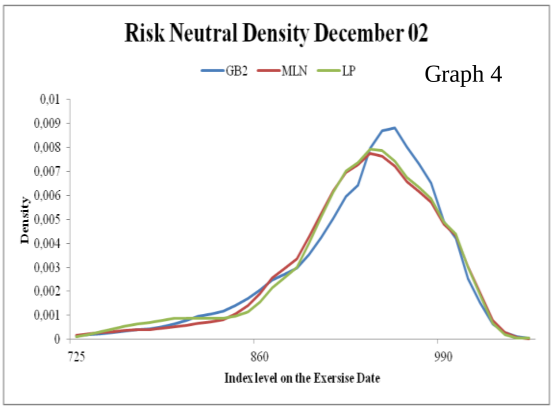
9. Appendices

The following graphs (graphs 1- 20) represent the quarterly RND estimated from quarterly out-of-the-money option prices before crisis.






Risk Neutral Density September 02
GB2 MLN LP
Graph 3
Density
0,01 0,009 0,008 0,007 0,006 0,005 0,004 0,003 0,002 0,001 0
675 900 1015
Index level on the Exersise Date


Risk Neutral Density December 02
GB2 MLN LP
Graph 4
Density
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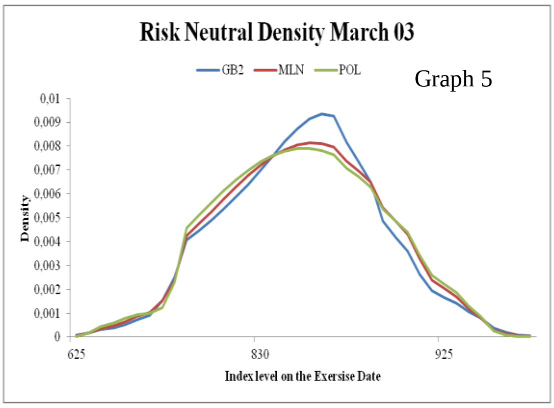
725 860 990
Index level on the Exersise Date


Risk Neutral Density March 03
GB2 MLN POL
Graph 5
Density
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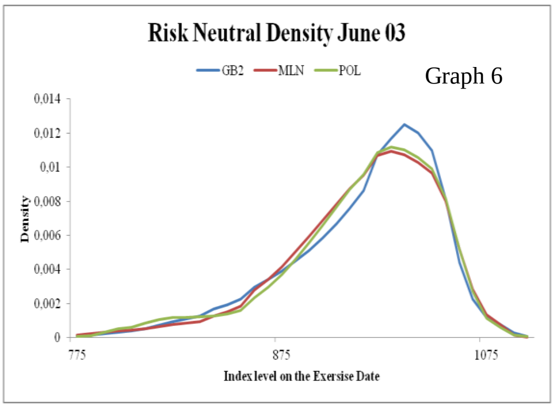
625 830 925
Index level on the Exersise Date


Risk Neutral Density June 03
GB2 MLN POL
Graph 6
Density
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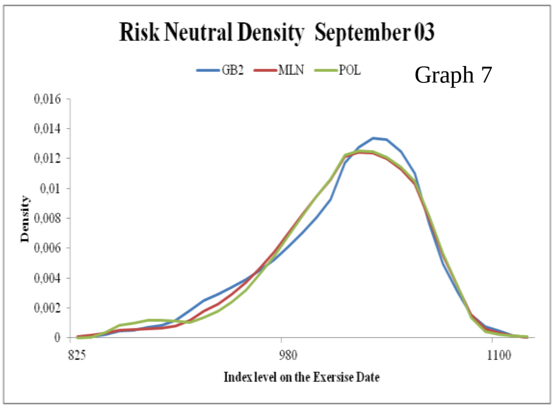
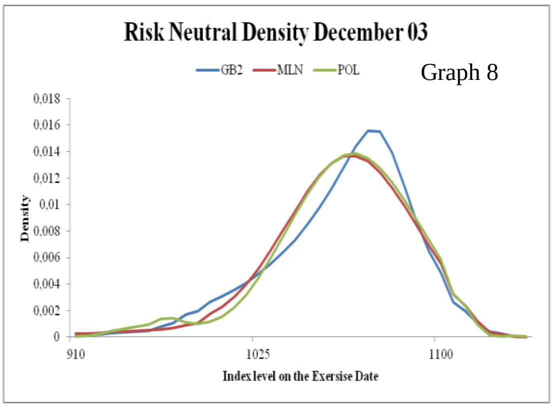
775 875 1075
Index level on the Exersise Date


Risk Neutral Density September 03
GB2 MLN POL
Graph 7
Density
0,016 0,014 0,012 0,01 0,008 0,006 0,004 0,002 0
825 980 1100
Index level on the Exersise Date


Risk Neutral Density December 03
GB2 MLN POL
Graph 8
Density
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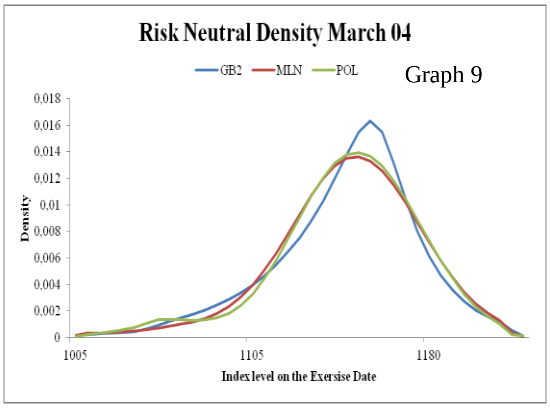
910 1025 1100
Index level on the Exersise Date


Risk Neutral Density March 04
GB2 MLN POL
Graph 9
Density
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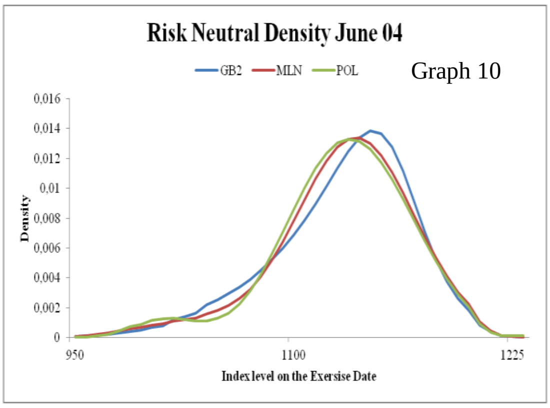
1005 1105 1180
Index level on the Exersise Date


Risk Neutral Density June 04
GB2 MLN POL
Graph 10
Density
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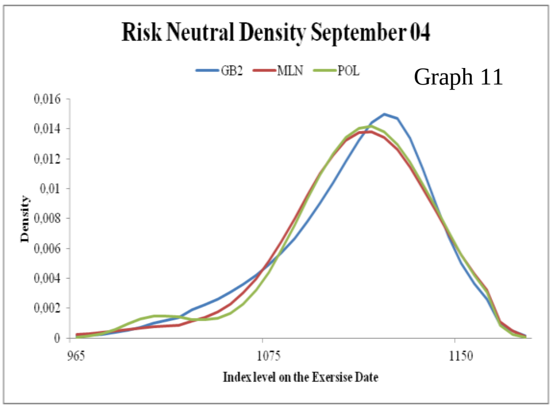
950 1100 1225
Index level on the Exersise Date


Risk Neutral Density September 04
GB2
MLN
POL
Graph 11
Density
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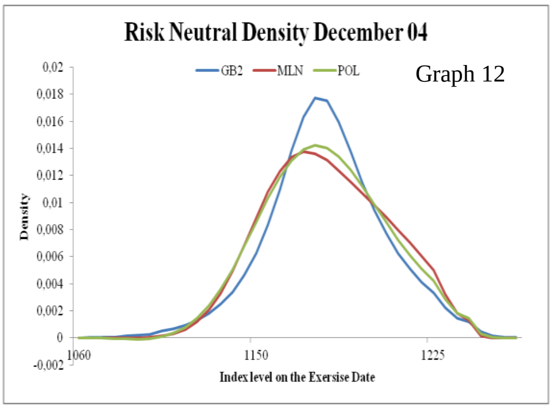
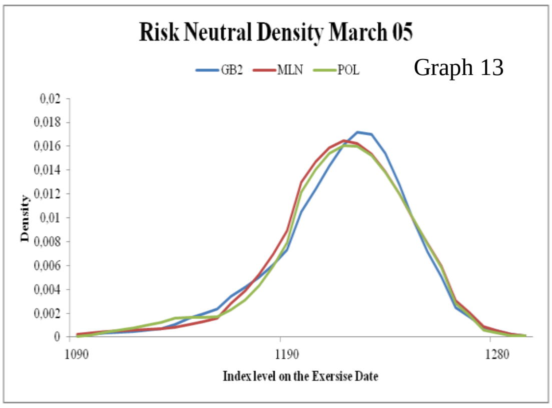
1075
1150
Index level on the Exersise Date


Risk Neutral Density December 04
GB2
MLN
POL
Graph 12
Density
1060
1150
1225
Index level on the Exersise Date


Risk Neutral Density March 05
GB2
MLN
POL
Graph 13
Density
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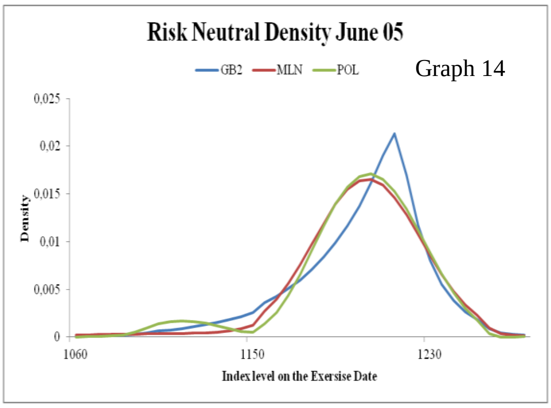
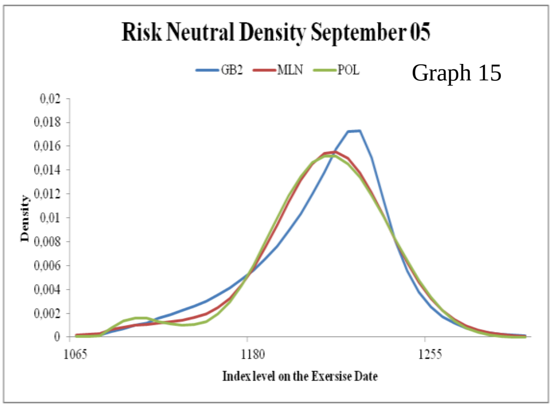
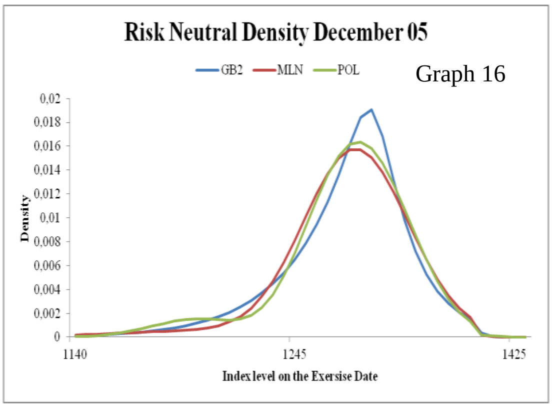
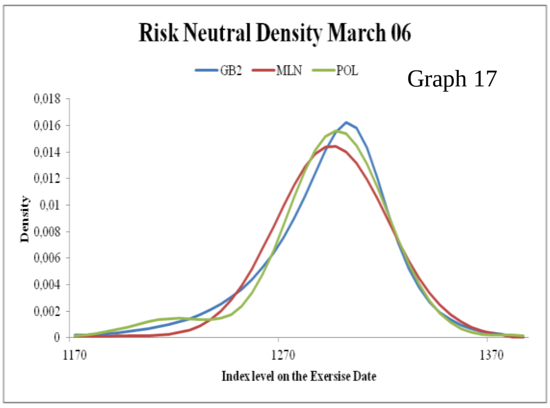
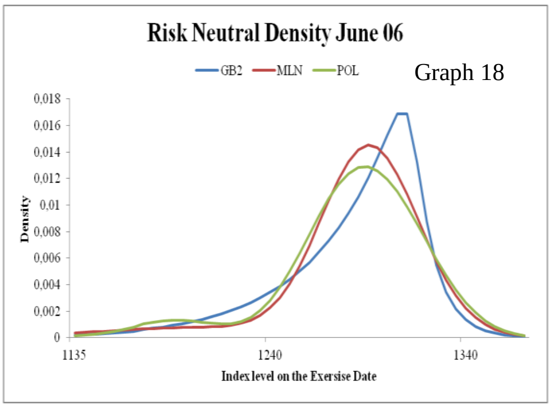
1190
1280
Index level on the Exersise Date


Risk Neutral Density June 05
GB2
MLN
POL
Graph 14
Density
1060
1150
1230
Index level on the Exersise Date


Risk Neutral Density September 05
GB2
MLN
POL
Graph 15
Density
1065
1180
1255
Index level on the Exersise Date


Risk Neutral Density December 05
GB2
MLN
POL
Graph 16
Density
1140
1245
1425
Index level on the Exersise Date


Risk Neutral Density March 06
GB2
MLN
POL
Graph 17
Density
1170
1270
1370
Index level on the Exersise Date


Risk Neutral Density June 06
GB2
MLN
POL
Graph 18
Density
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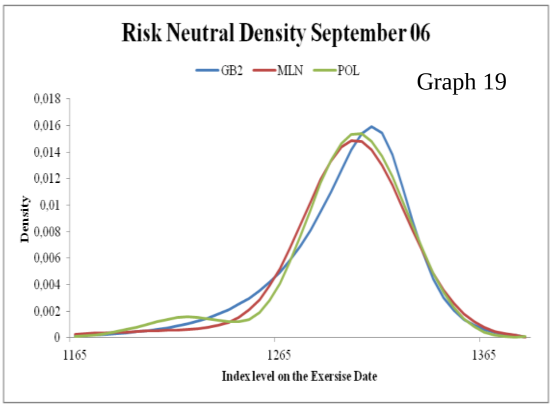
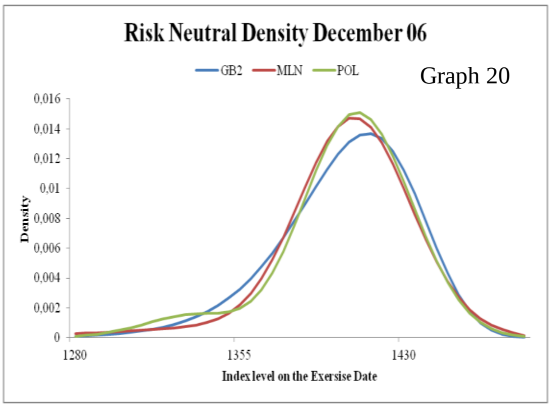
1240
1340
Index level on the Exersise Date





The tables 10 and 11 contain the RND descriptive statistics quarterly values for each of the parametric methods applied. The months marked with an asterisk correspond to a restricted RND estimated through the LP method. Likewise, two stars corresponds to RND with negative areas obtained through the LP method after applying Jondeau and Rockinger (2001) restrictions.

Table 10 – Quarterly RND statistics during the first half of the pre-crisis period.

	GB2	MLN	LP
Mar. 02			
G-Adjusted	0.0003	0.0002	0.0003
Total G	5.4896	2.5761	4.3269
Mean	1087.8038	1087.8284	1087.8286
Std. Dev.	56.4125	57.0474	56.2427
Skewness	-0.9349	-1.1492	-0.8317
Kurtosis	5.5641	5.9280	4.6956
JB Test	19.301	26.556	10.814
Jun. 02			
G-Adjusted	0.0002	0.0002	0.0002
Total G	3.0436	2.1453	3.1196
Mean	1065.8897	1065.8992	1065.8992
Std. Dev.	48.7760	48.3736	47.9418
Skewness	-0.8575	-0.9337	-0.6996
Kurtosis	5.6884	5.3451	4.4656
JB Test	25.846	22.841	10.435
* Sep. 02			
G-Adjusted	0.0003	0.0002	0.0004
Total G	4.7729	2.6316	9.6408
Mean	913.6582	914.0927	914.2235
Std. Dev.	66.1098	66.8847	66.9538
Skewness	-0.8717	-1.0361	-0.6768
Kurtosis	4.8898	5.0289	4.9592
JB Test	18.730	23.828	16.067
Dec. 02			
G-Adjusted	0.0003	0.0001	0.0002
Total G	4.5393	0.5000	2.9407
Mean	934.5644	934.5696	934.5697
Std. Dev.	61.6142	61.1475	61.0223
Skewness	-0.7634	-0.7568	-0.4852
Kurtosis	5.2028	4.9598	4.5007
JB Test	19.456	16.608	8.650
Mar. 03			
G-Adjusted	0.0003	0.0001	0.0002
Total G	3.1271	0.5800	1.0237
Mean	838.2554	838.3038	838.3037
Std. Dev.	60.1710	58.9390	58.9203
Skewness	-0.4583	-0.4709	-0.4681
Kurtosis	5.1813	4.0840	4.0645
JB Test	13.529	4.984	4.857

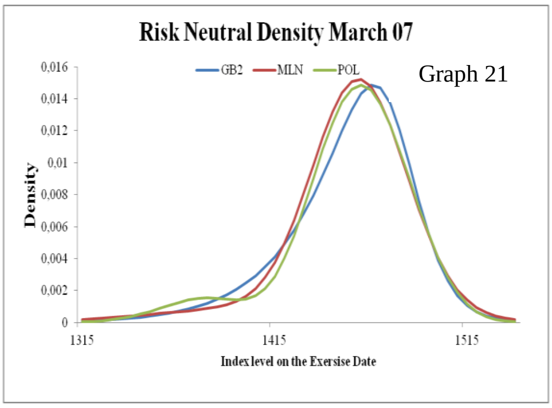
	GB2	MLN	LP
Jun. 03			
G-Adjusted	0.0001	0.0001	0.0002
Total G	1.1227	0.4836	1.3203
Mean	961.4480	961.4480	961.4481
Std. Dev.	43.5235	43.3337	43.1193
Skewness	-0.6196	-0.7596	-0.5075
Kurtosis	5.3750	5.1446	4.4844
JB Test	18.239	17.556	8.219
Sep. 03			
G-Adjusted	0.0004	0.0004	0.0004
Total G	7.2097	5.9325	6.8255
Mean	1004.9505	1004.9456	1004.9543
Std. Dev.	40.6830	41.1892	40.6879
Skewness	-0.6021	-0.7891	-0.5143
Kurtosis	5.4185	6.3957	5.0949
JB Test	14.599	28.044	10.894
Dec. 03			
G-Adjusted	0.0002	0.0001	0.0003
Total G	3.1486	1.3725	4.4171
Mean	1056.0811	1056.0811	1056.0822
Std. Dev.	37.6659	38.7438	37.6891
Skewness	-1.0336	-1.2217	-0.8233
Kurtosis	5.9777	7.4008	5.1353
JB Test	33.945	65.455	18.782
Mar. 04			
G-Adjusted	0.0001	0.0001	0.0001
Total G	0.9731	0.3127	1.1818
Mean	1142.1376	1142.1377	1142.1372
Std. Dev.	36.1380	36.1713	35.8664
Skewness	-0.7996	-0.9430	-0.6280
Kurtosis	5.8403	5.9515	4.8096
JB Test	27.889	32.205	12.737
* Jun. 04			
G-Adjusted	0.0001	0.0002	0.0003
Total G	1.0631	1.9701	5.1756
Mean	1118.2386	1118.2386	1118.2385
Std. Dev.	38.4243	38.6413	38.6499
Skewness	-1.1496	-1.0965	-0.6952
Kurtosis	5.8128	5.3693	5.0532
JB Test	35.195	27.794	16.396

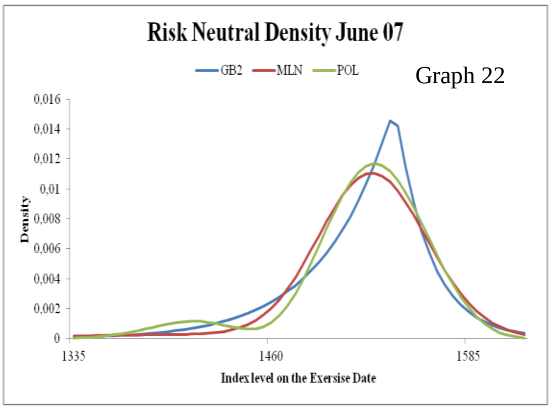
Table 11 – Quarterly RND statistics during the second half of the pre-crisis period.

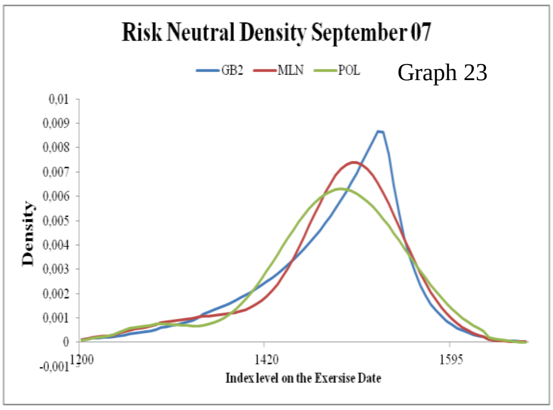
	GB2	MLN	LP
Sep. 04			
G-Adjusted	0.0002	0.0002	0.0002
Total G	1.9114	1.5675	2.7137
Mean	1105.5631	1105.5631	1105.5621
Std. Dev.	36.0410	36.3415	35.9726
Skewness	-1.0248	-1.1728	-0.8017
Kurtosis	5.7460	6.1460	4.8788
JB Test	27.396	35.932	14.235
** Dec. 04			
G-Adjusted	0.0006	0.0005	0.0005
Total G	29.8332	22.7902	21.6705
Mean	1181.1208	1181.1218	1181.1218
Std. Dev.	30.6736	28.9863	29.2551
Skewness	0.2009	0.1366	0.1578
Kurtosis	6.1946	3.8482	3.1626
JB Test	29.804	1.281	0.362
Mar. 05			
G-Adjusted	0.0001	0.0001	0.0002
Total G	1.3704	1.5146	2.3238
Mean	1209.6618	1252.1190	1209.6618
Std. Dev.	30.8438	31.6348	30.6112
Skewness	-0.9276	-1.0506	-0.7150
Kurtosis	5.6750	6.0631	4.5907
JB Test	25.609	33.344	11.057
* Jun. 05			
G-Adjusted	0.0003	0.0001	0.0003
Total G	7.9296	1.8557	2.3238
Mean	1197.2010	1197.2002	1197.2009
Std. Dev.	31.9898	33.7099	32.5933
Skewness	-1.2199	-1.8845	-0.8247
Kurtosis	6.5173	11.6265	5.9605
JB Test	51.155	247.402	32.062
* Sep. 05			
G-Adjusted	0.0002	0.0002	0.0002
Total G	3.5333	3.9993	5.7911
Mean	1203.8761	1203.8761	1203.8761
Std. Dev.	34.6237	34.8912	34.5605
Skewness	-1.2809	-1.3027	-0.8999
Kurtosis	6.4331	6.2702	5.1997
JB Test	51.989	49.534	22.887

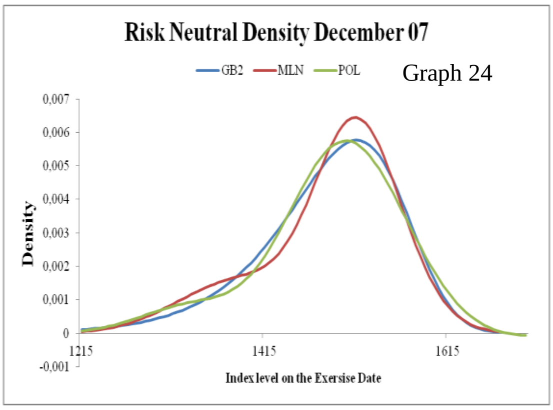
	GB2	MLN	LP
Dec. 05			
G-Adjusted	0.0002	0.0001	0.0002
Total G	3.1743	2.4184	3.6441
Mean	1267.3984	1267.3984	1267.3984
Std. Dev.	31.1387	31.5109	30.9197
Skewness	-0.8053	-1.0461	-0.6219
Kurtosis	5.9481	7.4122	4.9474
JB Test	34.326	72.528	16.241
Mar. 06			
G-Adjusted	0.0003	0.0002	0.0003
Total G	8.7109	3.2015	9.7903
Mean	1288.8215	1288.7928	1288.8229
Std. Dev.	32.3760	34.0712	32.4890
Skewness	-0.8580	-1.8580	-0.6201
Kurtosis	5.5468	14.0046	4.9760
JB Test	28.686	410.352	16.555
* Jun. 06			
G-Adjusted	0.0002	0.0002	0.0004
Total G	7.1479	5.7423	25.7014
Mean	1279.9111	1279.9111	1279.9111
Std. Dev.	39.5096	40.4966	39.8860
Skewness	-1.5642	-1.6995	-0.7655
Kurtosis	7.0034	7.8634	5.0233
JB Test	27.396	35.932	14.235
Sep. 06			
G-Adjusted	0.0002	0.0001	0.0002
Total G	3.3288	1.7978	4.4578
Mean	1294.7535	1294.7504	1294.7528
Std. Dev.	33.5319	34.2939	33.6853
Skewness	-1.1055	-1.3336	-0.8398
Kurtosis	5.8761	7.4789	5.0853
JB Test	39.483	81.524	21.509
Dec. 06			
G-Adjusted	0.0002	0.0001	0.0001
Total G	4.8196	2.7172	3.9092
Mean	1273.4754	1273.4734	1273.4758
Std. Dev.	30.1962	31.3004	30.7753
Skewness	-0.7869	-1.0900	-0.7480
Kurtosis	4.5269	6.8723	4.9161
JB Test	18.031	74.052	22.160

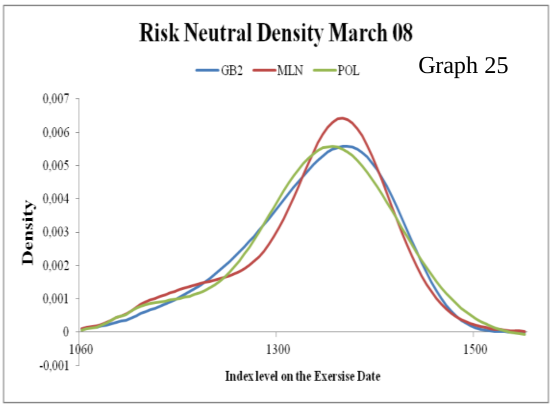
The subsequent graphs (graphs 21-40) represent the quarterly RND estimated from quarterly out-of-the-money option prices during the crisis periods.

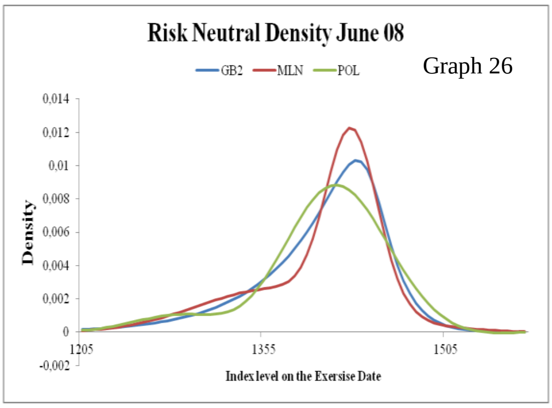


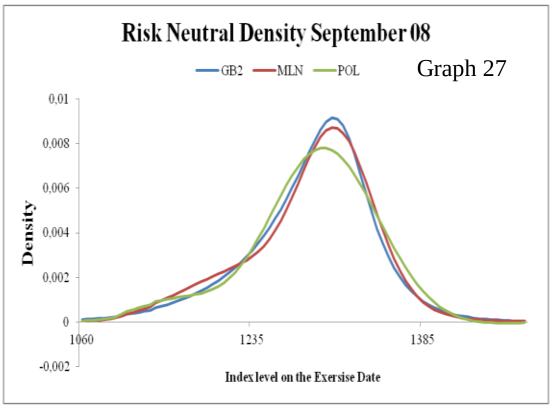


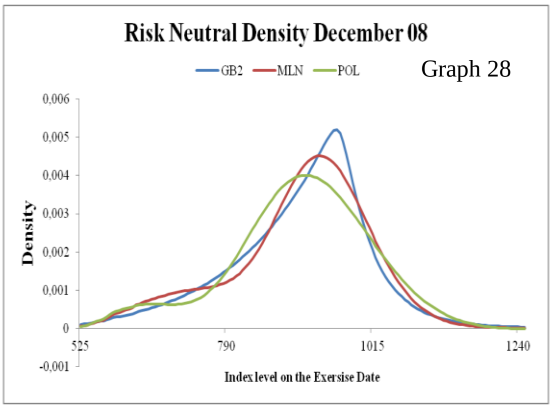


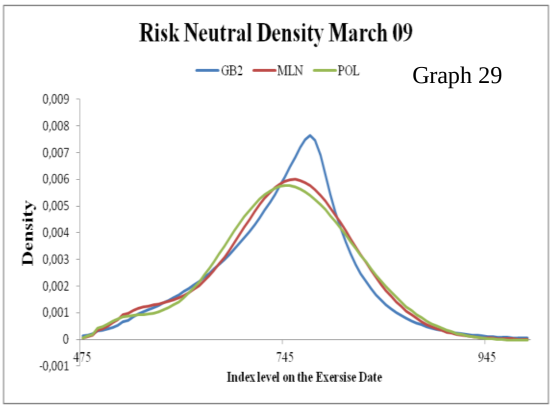











Risk Neutral Density March 09
GB2 MLN POL
Graph 29
Density
0,009
0,008
0,007
0,006
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0,004
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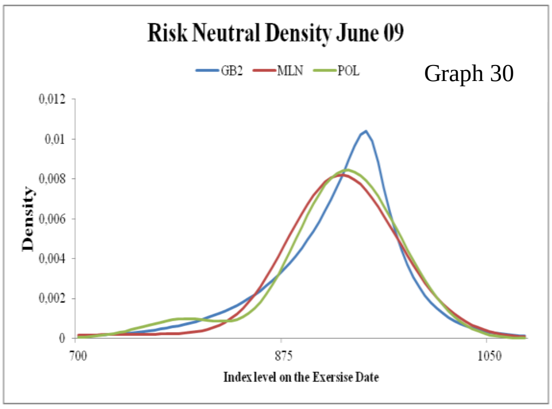
0,002
0,001
0
-0,001
475
745
945
Index level on the Exersise Date


Risk Neutral Density June 09
GB2 MLN POL
Graph 30
Density
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0
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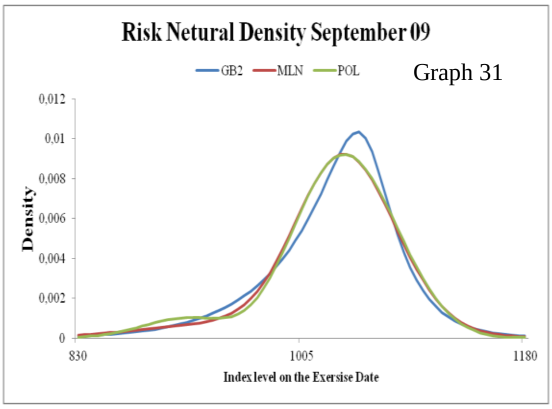
875
1050
Index level on the Exersise Date


Risk Netural Density September 09
GB2 MLN POL
Graph 31
Density
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0,01
0,008
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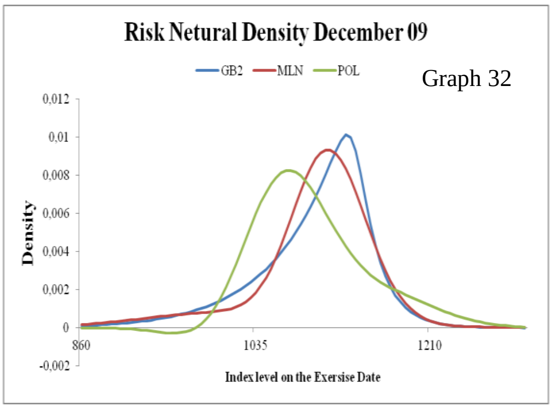
0,004
0,002
0
830
1005
1180
Index level on the Exersise Date


Risk Netural Density December 09
GB2 MLN POL
Graph 32
Density
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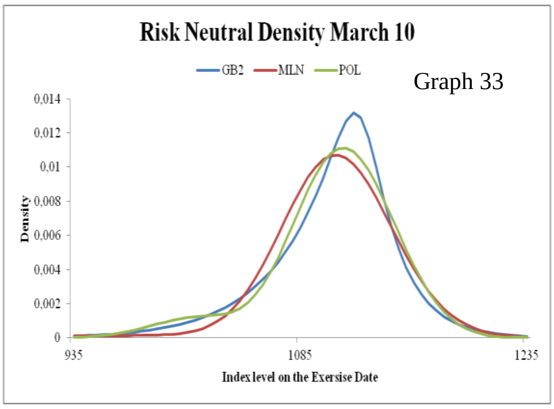
0,008
0,006
0,004
0,002
0
-0,002
860
1035
1210
Index level on the Exersise Date


Risk Neutral Density March 10
GB2 MLN POL
Graph 33
Density
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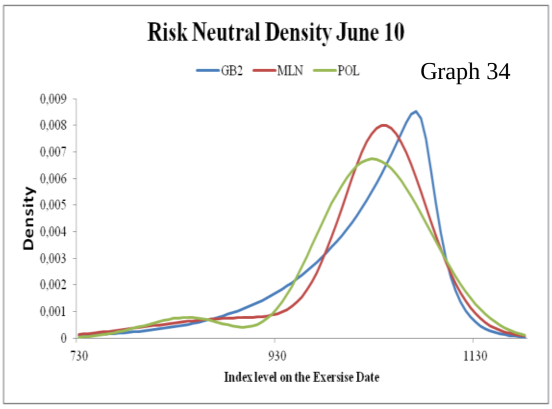
0,01
0,008
0,006
0,004
0,002
0
935
1085
1235
Index level on the Exersise Date


Risk Neutral Density June 10
GB2 MLN POL
Graph 34
Density
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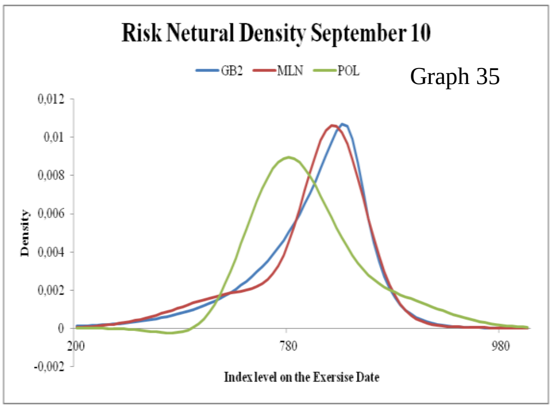
0,008
0,007
0,006
0,005
0,004
0,003
0,002
0,001
0
730
930
1130
Index level on the Exersise Date


Risk Netural Density September 10
GB2 MLN POL
Graph 35
Density
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0,008
0,006
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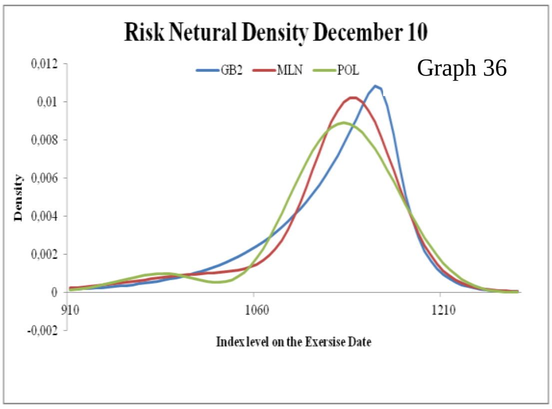
0
-0,002
200
780
980
Index level on the Exersise Date


Risk Netural Density December 10
GB2 MLN POL
Graph 36
Density
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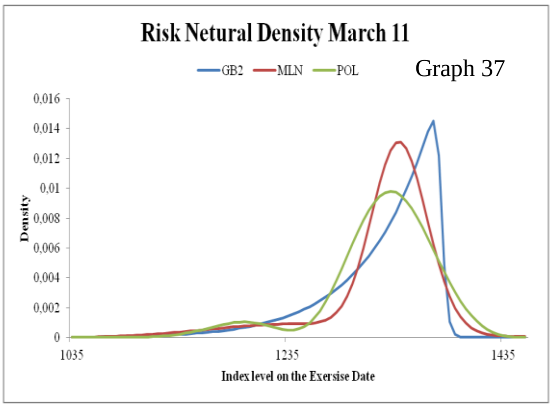
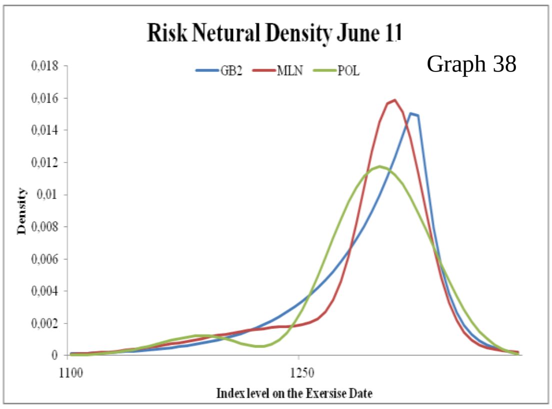
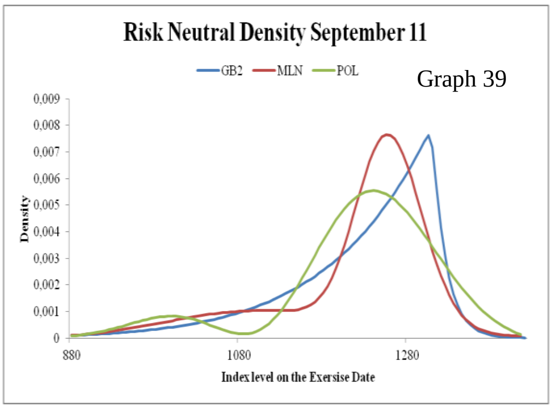
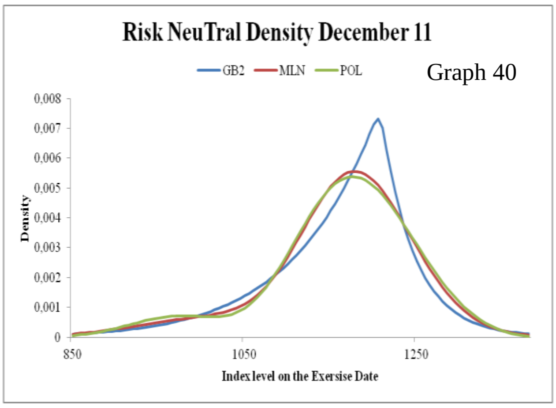
0,002
0
-0,002
910
1060
1210
Index level on the Exersise Date









Tables 12 and 13 enclose the RND descriptive statistics quarterly values for each of the parametric methods applied. The months marked with a star correspond to a restricted RND estimated through the LP method. Likewise, two asterisks corresponds to RND with negative areas obtained through the LP method after applying Jondeau and Rockinger (2001) restrictions.

Table 12 – Quarterly RND statistics during the first half of the crisis period.

		GB2	MLN	LP
	Mar. 07			
	G-Adjusted	0.0001	0.0001	0.0002
	Total G	2.7726	2.6338	3.2789
	Mean	1451.3162	1451.2543	1451.3152
	Std. Dev.	33.8209	35.0865	33.9073
	Skewness	-1.0701	-1.0777	-0.8108
	Kurtosis	5.5804	6.2761	4.7512
	JB Test	44.954	61.515	22.784
	Jun. 07			
	G-Adjusted	0.0002	0.0001	0.0003
	Total G	12.2215	4.2962	16.2319
	Mean	1515.2155	1515.3309	1513.9346
	Std. Dev.	45.6733	46.8772	45.3754
	Skewness	-1.0578	-1.5939	-0.9212
	Kurtosis	6.1806	9.3009	5.2821
	JB Test	57.759	197.376	34.049
**	Mar. 08			
	G-Adjusted	0.0003	0.0004	0.0004
	Total G	25.9003	41.5657	36.4180
	Mean	1331.9374	1330.9067	1331.9385
	Std. Dev.	81.1874	82.9514	79.6774
	Skewness	-0.9328	-0.8548	-0.9137
	Kurtosis	4.4793	3.7414	3.0061
	JB Test	29.290	17.940	17.254
**	Jun. 08			
	G-Adjusted	0.0003	0.0003	0.0004
	Total G	17.1483	16.7000	36.1887
	Mean	1400.4788	1400.4788	1400.4787
	Std. Dev.	55.7389	54.9384	54.9366
	Skewness	-1.3305	-0.9353	-0.9068
	Kurtosis	6.2034	4.2276	4.1837
	JB Test	80.932	23.361	21.889
**	Mar. 09			
	G-Adjusted	0.0006	0.0003	0.0004
	Total G	27.3878	8.0783	12.0477
	Mean	735.4236	735.4239	735.4242
	Std. Dev.	80.4308	78.3019	78.2554
	Skewness	-0.6358	-0.5420	-0.6076
	Kurtosis	5.0144	3.3843	3.3741
	JB Test	35.941	8.379	10.240

		GB2	MLN	LP
**	Sep. 07			
	G-Adjusted	0.0001	0.0002	0.0006
	Total G	2.0798	8.4738	69.7651
	Mean	1473.0353	1473.2881	1472.1581
	Std. Dev.	79.0080	78.1777	77.5359
	Skewness	-1.4030	-1.1654	-0.9255
	Kurtosis	6.2462	4.6066	4.2029
	JB Test	79.017	34.393	20.913
**	Dec. 07			
	G-Adjusted	0.0002	0.0001	0.0003
	Total G	10.4949	5.7941	15.1073
	Mean	1481.6376	1481.6376	1481.6538
	Std. Dev.	79.1919	78.2273	77.7186
	Skewness	-0.9310	-0.7445	-0.9326
	Kurtosis	4.5771	3.3592	3.0963
	JB Test	33.740	13.294	19.769
**	Sep. 08			
	G-Adjusted	0.0003	0.0002	0.0003
	Total G	13.1699	7.0918	12.0354
	Mean	1282.9059	1282.9060	1282.9061
	Std. Dev.	60.2283	58.7686	58.5919
	Skewness	-0.9260	-0.6737	-0.8044
	Kurtosis	5.5506	3.6761	3.5671
	JB Test	51.335	11.741	15.033
**	Dec. 08			
	G-Adjusted	0.0006	0.0006	0.0008
	Total G	52.3803	48.2572	111.6960
	Mean	897.4470	897.4480	897.4478
	Std. Dev.	115.7522	113.4152	402.6294
	Skewness	-0.7706	-0.7131	-1.1302
	Kurtosis	4.6932	3.4655	1.3777
	JB Test	43.464	18.661	64.185
	Sep. 09			
	G-Adjusted	0.0002	0.0003	0.0003
	Total G	7.5003	10.4230	14.3368
	Mean	1029.1103	1029.3188	1029.1103
	Std. Dev.	53.6559	53.6517	53.1965
	Skewness	-0.8358	-0.8818	-0.6611
	Kurtosis	5.4538	5.2887	4.4969
	JB Test	55.096	52.179	24.931

Table 13 – Quarterly RND statistics during the second half of the crisis period.

	GB2	MLN	LP
Jun. 09			
G-Adjusted	0.0005	0.0004	0.0008
Total G	32.6804	14.8231	34.9146
Mean	919.2203	919.2203	919.2210
Std. Dev.	59.2896	60.5329	58.6444
Skewness	-0.8629	-1.2500	-0.6836
Kurtosis	5.5388	7.8612	4.5556
JB Test	62.039	196.720	28.236
Mar. 10			
G-Adjusted	0.0003	0.0003	0.0004
Total G	13.6522	11.4519	18.4217
Mean	1104.6343	1104.6324	1104.6352
Std. Dev.	43.8943	45.0731	43.0140
Skewness	-0.8634	-1.3507	-0.6607
Kurtosis	5.7385	9.8366	4.2465
JB Test	69.875	360.247	21.998
* Jun. 10			
G-Adjusted	0.0004	0.0004	0.0009
Total G	36.8091	28.5667	165.3755
Mean	1090.0284	1090.0284	1090.0287
Std. Dev.	76.3380	77.3764	77.2265
Skewness	-1.4238	-1.3887	-0.6595
Kurtosis	5.9671	5.7909	5.2548
JB Test	111.344	102.064	44.923
* Mar. 11			
G-Adjusted	0.0004	0.0003	0.0007
Total G	48.0905	30.5449	122.9449
Mean	1320.8239	1320.8239	1320.8239
Std. Dev.	53.8601	55.3376	54.6380
Skewness	-1.7519	-1.7876	-0.8200
Kurtosis	7.2588	7.4787	5.4575
JB Test	210.364	227.147	60.376
* Jun. 11			
G-Adjusted	0.0002	0.0002	0.0005
Total G	16.8954	16.4397	42.5336
Mean	1332.4597	1332.4597	1332.4610
Std. Dev.	45.7852	45.9258	46.0551
Skewness	-1.4965	-1.3629	-0.8563
Kurtosis	6.8109	5.9120	5.5321
JB Test	165.349	112.032	65.800

	GB2	MLN	LP
** Dec. 09			
G-Adjusted	0.0004	0.0005	0.0022
Total G	29.1973	32.5379	926.9910
Mean	1092.3852	1092.3852	1092.3852
Std. Dev.	61.1924	61.9916	54.7303
Skewness	-1.2524	-1.2673	1.1075
Kurtosis	5.9505	6.0801	3.2851
JB Test	101.739	108.063	33.873
** Sep. 10			
G-Adjusted	0.0003	0.0003	0.0020
Total G	13.3782	14.3698	710.6745
Mean	1064.6673	1064.6673	1064.6674
Std. Dev.	56.7139	55.7456	51.4659
Skewness	-1.2754	-1.0464	0.1263
Kurtosis	6.0034	4.3503	3.5774
JB Test	97.687	39.026	2.499
** Dec. 10			
G-Adjusted	0.0003	0.0003	0.0005
Total G	16.7527	15.7396	51.8872
Mean	1190.3642	1190.3642	1190.3643
Std. Dev.	58.8232	59.4165	59.2705
Skewness	-1.3996	-1.3139	-0.7823
Kurtosis	6.2908	5.8167	5.2637
JB Test	122.877	97.692	49.851
Sep. 11			
G-Adjusted	0.0006	0.0006	0.0009
Total G	96.3723	76.6909	195.1796
Mean	1178.0065	1178.0065	1178.0055
Std. Dev.	95.3579	95.3381	94.1866
Skewness	-1.4868	-1.2682	-1.4622
Kurtosis	5.8958	4.7453	4.5125
JB Test	132.079	72.677	83.104
Dec. 11			
G-Adjusted	0.0004	0.0003	0.0004
Total G	36.0885	24.0341	28.3609
Mean	1159.6329	1159.6329	1159.6321
Std. Dev.	90.2832	89.2463	88.5639
Skewness	-0.9377	-0.8313	-0.8055
Kurtosis	5.4302	4.3232	3.9210
JB Test	74.993	35.935	27.405

Table 14 – Pre-crisis monthly samples composition.

.

Period	Number of Puts	Number of Calls	Period	Number of Puts	Number of Calls
Jan. 03	18	21	Jan. 05	43	17
Feb. 03	17	25	Feb. 05	31	20
Mar. 03	20	37	Mar. 05	39	18
Apr. 03	18	20	Apr. 05	29	25
May 03	22	18	May 05	17	46
Jun. 03	27	33	Jun. 05	24	42
July 03	26	15	July 05	32	23
Aug. 03	31	11	Aug. 05	38	16
Sep. 03	28	19	Sep. 05	39	28
Oct. 03	25	29	Oct. 05	40	20
Nov. 03	35	16	Nov. 05	36	24
Dec. 03	35	26	Dec. 05	52	20
Jan. 04	36	13	Jan. 06	36	24
Feb. 04	36	19	Feb. 06	38	21
Mar. 04	37	25	Mar. 06	49	23
Apr. 04	28	20	Apr. 06	40	26
May 04	24	22	May 06	42	21
Jun. 04	22	41	Jun. 06	54	29
July 04	38	18	July 06	46	33
Aug. 04	34	22	Aug. 06	48	22
Sep. 04	16	39	Sep. 06	48	23
Oct. 04	33	20	Oct. 06	56	15
Nov. 04	36	14	Nov. 06	59	19
Dec. 04	30	38	Dec. 06	69	20

	Puts	Calls
Mean	34.94	23.67
Monthly Mean		58.60

Table 14 presents the average number of out-of-the-money put and call options contracts included within the pre-crisis monthly samples. The monthly mean indicates the average number of observations included in each month within the pre-crisis period.

Table 15 - Monthly samples composition during crisis period.

Period	Number of Puts	Number of Calls	Period	Number of Puts	Number of Calls
Jan. 07	60	19	July 09	90	42
Feb. 07	50	23	Aug. 09	92	35
Mar. 07	72	23	Sep. 09	96	53
Apr. 07	58	29	Oct. 09	105	44
May 07	65	18	Nov. 09	104	63
Jun. 07	72	22	Dec. 09	101	61
July 07	54	33	Jan. 10	116	46
Aug. 07	68	23	Feb. 10	108	64
Sep. 07	58	44	Mar. 10	85	74
Oct. 07	69	27	Apr. 10	115	39
Nov. 07	66	34	May 10	112	46
Dec. 07	78	57	Jun. 10	87	70
Jan. 08	90	45	July 10	100	71
Feb. 08	58	58	Aug. 10	100	53
Mar. 08	55	68	Sep. 10	93	57
Apr. 08	63	53	Oct. 10	114	42
May 08	70	38	Nov. 10	111	38
Jun. 08	65	46	Dec. 10	108	49
July 08	42	64	Jan. 11	115	44
Aug. 08	43	66	Feb. 11	114	42
Sep. 08	55	68	Mar. 11	124	41
Oct. 08	58	58	Apr. 11	104	47
Nov. 08	90	105	May 11	111	31
Dec. 08	83	115	Jun. 11	119	49
Jan. 09	71	78	July 11	95	64
Feb. 09	61	77	Aug. 11	100	47
Mar. 09	48	103	Sep. 11	104	79
Apr. 09	73	56	Oct. 11	101	76
May 09	86	44	Nov. 11	139	34
Jun. 09	92	65	Dec. 11	99	91

	Puts	Calls
Mean	85.58	52.52
Monthly Mean		138.1

Table 15 presents the average number of out-of-the-money put and call options contracts included within the crisis monthly samples. The annual average indicates the average number of observations included in each sample.

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