Riders on the Storm

A risk preference profiling on investors who rode the subprime financial storm

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Abstract

This dissertation estimates risk-neutral densities from 3-week contracts on the S&P 500 index in an attempt to characterize the underlying index in a risk-neutral environment through the statistics derived from the implied distributions for two samples: pre-subprime-crisis and crisis. The distributions are estimated using mixture of lognormal densities, generalized beta distribution of the second kind and lognormal-polynomials. The mean values are similar in the three methods employed, along with the standard deviation. Moreover, the distributions tend to be negatively skewed and leptokurtic for both samples.

The constant relative risk aversion coefficient is estimated for both samples assuming the power utility is well representative of investors' behavior. The method employed was the mixture of lognormal distributions under both expected utility (EU) and rank-dependent utility assumptions (RDEU). The obtained coefficients for the pre-crisis sample were: 2,81 (EU) and 4,41 (RDEU) while in the crisis sample, the coefficients obtained were: 0,47 (EU) and -1,94 (RDEU). In line with literature, by applying the real-world transformation (RDEU) to the mixture of lognormal distribution estimated RND produced distributions with higher mean, lower standard deviation, less negatively skewed and with lower Kurtosis

Nesta dissertação, estimam-se *risk-neutral densities* a partir de derivados sobre o índice S&P 500 para dois períodos: pré-crise e crise. Com o objetivo de descrever o impacto da crise do *subprime* no mercado americano. As distribuições são extraídas usando três métodos diferentes: mistura de distribuições log-normais, distribuição beta generalizada do segundo tipo e log-normal-polinomiais. Da aplicação das três metodologias obtêm-se médias e desvios-padrão semelhantes. As distribuições obtidas tendem a ter *skewness* negativa e um valor de *kurtosis* superior a 3.

O coeficiente de aversão ao risco é estimado para ambos os períodos assumindo que a função de utilidade representativa é a *power utility*. O método utilizado foi a mistura de log-normais assumindo *Expected utility* (EU) e *rank-dependent expected utility* (RDEU). Os coeficientes estimados foram de 2,81 (EU) e 4,41 (RDEU) para o período pré-crise. Para o período de crise os valores obtidos foram de 0,47 (EU) e -1,94 (RDEU). Finalmente, ao transformar as *Risk-neutral densities* em *Real-world densities*, obtem-se distribuições com média mais elevada, desvio-padrão mais baixo, *skewness* menos negativa e *kurtosis* mais baixa, estando estas conclusões de acordo com o exposto na literatura.

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Introduction

What will tomorrow's price be? This is the billion-dollar question everyone in the financial world is trying to answer. Firstly, the analysts based their predictions in fundamental analysis (DCF's, Ratios, multiples, among other methods). With the increased impact of the financial sector in society overall, the study of the evolution of an asset's price drawn more attention. Nowadays, practitioners and researchers seek to obtain the real-world densities describing the probability distribution of a given price for the future, since the "standard" distributions such as the normal or the lognormal, for returns and prices respectively do not describe well enough the real-world behavior.

With the development of technologies and with more active derivatives market, practitioners seek to derive the implied market densities from option prices, as these provide fruitful information regarding not only the mean and the volatility level expected by the market, but also regarding the skewness and kurtosis. The implied market densities may be used to support monetary policy decisions, price complex derivatives, improve VaR measurements or assess the impact of a given event and represent an essential tool for regulators and practitioners.

The ability to forecast is even more important in times of financial turmoil as markets behave in an erratic and apparently less rational way. In this sense, I estimate the risk neutral densities for options on the S&P 500 index during the financial crisis of 2008 using three different methodologies: Mixture of Lognormal Distributions (MLN), Generalized Beta Distribution of the second kind (GB2) and lognormal-polynomials (Polynomials). Moreover, I estimate the risk aversion coefficients under the expected utility (EU) and rank-dependent expected utility (RDEU) and obtain the real-world densities (RWD). Besides drawing conclusions from the data obtained, I seek to compare how different models handle the task at hands. Moreover, I present a generalized description of the densities characterizing the market in a pre-crisis period and compare them with similar densities estimated during the crisis period, both in a risk neutral and real-world setting.

The results obtained point towards a decrease in risk aversion. Even if counterintuitive, Bliss and Panigirtzoglou (2004) indicate that the risk aversion tends to decrease with higher volatility levels in the market while Jackwerth (2000) and Liu et al. (2007) present further arguments supporting the findings obtained drawn from their studies on risk preferences in crisis periods.

Section 2 presents a broad description of the academic literature available on the methods to estimate RND and RWD. Section 3 identifies the data considered in this dissertation. Section 4 presents the methodologies employed. Section 5 presents empirical findings and comments on the relevant information. Section 6 includes the conclusion.

Literature Review

Risk Neutral Densities

Following the arbitrage pricing theory (Ross, 1976), the price of a given asset is the present value of its expected payoff discounted by the interest rate, i.e. the risk-free rate. Furthermore, under the assumption of a risk-neutral world, Ross (1976) obtains a complete risk-neutral density (RND) from the prices of European options. The resulting distribution is later defined by Brenden & Litzenberger (1978) as the state-contingent price of the future option price expiring at-the-money (Jackwerth, 1999):

$$RND = e^{rT} * \frac{\partial^2 c(x)}{\partial x^2}$$

According to Monteiro et al. (2008), the RND provides information about the probability distribution of the future price of the underlying asset, being a forward-looking measure of the risk-neutral market expectations. This is according to Pérignon & Villa (2002) one of the drivers for the shift of focus from implied volatility to RND study.

Bahra (1997) extends the research on the RND focusing on the conditions and assumptions taken for the function to be considered a density function. Apart from the assumption that investors are risk-neutral, the estimated probabilities must be strictly positive; the integration of the RND function for a set of all feasible exercise prices i.e. $x \geq 0$, must add to one and the market must be complete, perfect and free from arbitrage opportunities (Jackwerth, 1999). Moreover, Bahra (1997) states that the function must be continuous, convex and monotone for the equation presented above, to represent a RND.

Bellow this paper will further summarize the different methods described in the literature to estimate RNDs through parametric or non-parametric methods as well as the underlying models.

Parametric Methods

This approach states that RND follows a process that can be described by a given set of parameters, making assumptions regarding the characteristics that the model emulates. The fact that (more) assumptions are taken when compared to non-parametric methods along with its relative simplicity leads to easier estimation and faster computing, nevertheless, the methods are restricted by processes that allow for closed form solutions.

The literature describes different methods for estimating risk-neutral densities. Jackwerth (1999) divides parametric methods into three major categories: Mixture Methods, Generalized Methods, Expansion Methods and Stochastic Volatility Methods.

Mixture Methods

Mixture methods are supported by the fact that using two or more distributions to estimate the RND will yield a higher degree of flexibility when compared to a single lognormal by allowing for a more detailed parametrization (Jackwerth, 1999). This method consists in the weighted average of k-individual distributions, being the parameters estimated collectively.

The literature focusses mainly in using lognormal distributions when employing this method to estimate RNDs. Ritchey (1990) proposes a mixture of two lognormal distributions to study the non-normality of stock returns using option prices. Melick and Thomas (1997) use a mixture of three lognormal distributions to derive boundaries for American option prices on the crude oil futures, accounting for the possibility of early exercise during the Persian Gulf crisis. Moreover, proves that mixture of three lognormal distributions also performs better than the single lognormal model when representing the market expectations for a disruption in oil prices. The method was also applied to the extraction of RND on equity indexes Anagnou et al. (2002), to exchange rates derivatives by Campa et al (1998) and Jondeau and Rockinger (2000) and finally to interest rate derivatives by Bahra (1997) and Coutand et al (2001).

Mixture methods are associated with relatively easy application, yield non-negative densities and its parameters are informative. However, Anagnou et al (2002) report that the method is somewhat inaccurate to represent densities extracted for equity indexes.

Generalized Distributions

An intuitive insight over generalized distributions is to consider them originating distributions whose special cases are well known distributions such as the lognormal, gamma or even the exponential distribution. Bookstaber and McDonald (1987) introduced the generalized beta distribution of the 2nd kind (GB2). This distribution is referred to as a "descriptive tool rather than as a definitive distribution" (Bookstaber and McDonald, 1987) when applied to estimating probability density functions for stock returns in the United States. The GB2 is characterized by 4 parameters and is able to capture the effects of the skewness and kurtosis (Taylor,2005). Moreover, the distribution is characterized by its flexibility and by fulfilling the non-negativity density condition. While its parameters have no direct individual interpretation, there is a large set of well-known special cases such as lognormal distribution (when $a \to 0$ and $q \to \infty$) or the Burr-III distribution (when $q = 1$) that arise from the GB2, hence its flexibility in characterizing the shape and moments of the RND.

Anagnou et al (2002) apply the GB2 to estimate the RND using option over the US Dollar/Sterling Pound exchange rate and over the S&P500 Index while Sherrick, Garcia and Tirupattur (1996) use the Burr III distribution to estimate the RND on soybean futures.

Expansion Methods

The logic behind expansion methods consists in considering a distribution that is fairly representative of the density of asset price then add correction terms (i.e. expansion of a Taylor series) to increase flexibility and obtain a more representative RND distribution. Lognormal or normal distributions are recurrently used in the literature to represent the underlying prices and returns respectively. Jondeau and Rockinger (2001) use Gram-Charlier expansions to estimate the RND on options over the French Franc/Deutsch Mark exchange rate, while Rubinstein (1998) uses an Edgeworth expansion to obtain RND from both European and America options. Additionally, Madan and Milne (1994) use a series of hermite polynomials to expand a lognormal distribution. The solution presented is derived by the assumption that the density of standardized returns is represented by a standardized normal distribution multiplied by a general function that is approximated by the series of hermite polynomials. The authors use this approach in the context of contingent-claim valuation and first introduce the lognormal-polynomial density function as a way to recover RND from option prices. Abken et al (1996) proceed the development of such method and apply to the study of Eurodollar futures' RND.

This method is criticized by not guaranteeing non-negativity (Jackwerth, 1999), nevertheless it captures skewness and kurtosis better than a simple lognormal distribution (Madan and Milne, 1994).

Stochastic Volatility method

Stochastic volatility methods consider that the process followed by the price of the underlying asset is fully specified and that a realistic specification should consider the stochastic volatility. Hull and White (1987) apply this method to European stock options while Melino & Turnbull (1990) apply it to foreign exchange currency options. Ball & Roma (1994) extend their research covering the bias that the Black-Scholes pricing formula presents when compared to the inclusion of stochastic volatility.

Heston (1993) assumes that the stochastic volatility follows a diffusion process and derives a closed form solution for the pricing of a European call option considering stochastic volatility for the underlying asset. Conversely, Jorion (1989) firstly proposed that the fluctuations of the first and second moments of the distribution should be accounted for. Analogously, Jondeu and Rockinger (2000) refer to the price process as Geometric Browninan Motion along with a poisson process to account for the time varying feature of the distribution. Finally, Ball & Torous (1983) simplify this process by considering that there would only be one jump process during the life of the option on equity stocks.

Non-parametric methods

Conversely to parametric methods, the non-parametric methods do not make any assumption regarding the process followed by the price of the underlying asset. This means that this class of methods allows for more flexibility regarding the shape of the distribution, being that this method is especially appropriate when the underlying process changes with time (Monteiro et al, 2008).

This flexibility does come at a cost. Firstly, this class of methods is dependent on an exhaustive data set to estimate RNDs. According to Jackwerth (2004), this is because the number of variables to be estimated is much larger than in the parametric model. Moreover, the non-negativity condition and its integration to one, necessary for the extracted densities to be valid are not always assured by non-parametric methods (Jackwerth, 2004).

Kernel Regression

This method tries to fit a function to the observed data without specifying the form of the function to be estimated. Firstly, it is assumed that the observed implied volatility along with the respective strike price is the central point of a channel where the true value of the volatility sits. The further away from that reference point, the less likely it is for the function to cross, therefore the relative weight the data point has in the RND is also lower. This method is data-intensive and the trade-off between smoothness and data-fitting imposed by the amplitude of the kernel channel tends to create problems when fitting data with large gaps between strike prices (Jackwerth, 2004). Moreover, this method assumes that the Implied Volatility function remains stable during the estimation period (Taylor, 2005), which is somewhat questionable. Nevertheless, Aït-Sahalia and Lo (2000) apply kernel regressions to estimate RND on option over the S&P 500 index and Pritsker (1998) applies it on interest rates derivatives. Finally, despite the method sometimes producing negative densities, Aït-Sahalia and Duarte (2003) propose a modified kernel regression that enforces positive densities when extracting RND.

Maximum Entropy Methods

The principle behind Maximum entropy methods states that for a given set of testable information (I.e. the option prices in this study), the density that best reflects the current state of information is the one with largest entropy. Firstly, proposed in physics, the concept was broadly applied to information theory and statistics. Buchen and Kelly (1996) study its implementation to estimate RNDs from option prices, concluding that this method "is the least committal with respect to unknown or missing information" (Buchen and Kelly, 1996) being able to accurately fit a density given a broad set of strike prices and assuming that the observed prices match the theoretical call price formula (Taylor, 2005). Jackwerth and Rubinstein (1996) use the Maximum entropy method to estimate RND over S&P 500 European call options.

Curve Fitting Methods

This class of methods consists in fitting a given flexible function to RND or to implied volatilities (Jackwerth, 1999). The methods used range from simple polynomials to more complex spline interpolation (i.e. piecewise function constituted by the aggregation multiple polynomials) with the goal of increasing smoothness of the curve. Shimko (1993) firstly introduces this method by fitting a quadratic polynomial to the volatility smile. The obtained implied volatilities are then converted to option prices to finally estimate RND following Breenden and Litzenberger (1978) procedure. The fitting of such functions to the smile curve has close form solution, however, it may generate negative implied volatilities. In this sense, Rosenberg and Engle (1997) fit polynomials to the log of the volatility smile curve to prevent negative implied volatilities. Likewise, Jackwerth (2000) seeks to minimize the curvature of the smile curve and concludes that the obtained densities are non-negative. Conversely, Malz (1997) derives the implied volatility function through option deltas (i.e. first derivative of Black-Scholes call pricing function with respect to the underlying price) instead of using call option prices.

In contrast to fitting the function to volatility smile, Rubinstein (1994) directly applies curve fitting methods to estimate RND. In the same fashion, Mayhey (1995) maximizes the smoothness of the RND by directly approximating cubic splines.

Positive Convolution method

Bondarenko (2003) introduces this method with inherent characteristics of both parametric and non-parametric methods. The method consists in a mixture of normal densities with equispaced means and matched standard deviations. The author describes this method as a combination of a positive kernel with a different density and states that this method, despite simple to compute, avoids overfitting of data and guarantees arbitrage-free estimators (Bondarenko, 2003).

Risk Aversion

Inference from the application of the methods above described, regarding the future density of the underlying asset, would imply the assumption that investors are risk neutral as the output obtained is the risk neutral distribution. Bliss and Panigirtzoglou (2004) pose that this assumption is not reasonable as the RND does not reflect the invertors' behaviour relative to their risk tolerance and the marginal premium demanded by bearing more risk. Anagnou et al. (2002), Bliss and Panigirtzoglou (2004) and Liu et al (2007) support this conclusion due to the existence of positive relative risk aversion (RRA).

In this sense, a correction must be made if inference is to be taken from the information provided by RND (i.e. to obtain real-world densities). To obtain RWD, we need a function that faithfully represents the investor utility function. Moreover, we also need to derive that investor's risk aversion coefficient as each agent perceives risk in a different manner, being its actions under risky situations conditioned by their perception (Perignon and Villa, 2002).

Jackwerth (2000) proposes the abovementioned relation to assume the form of:

$$RND = RWD\ X\ Risk\ Aversion\ Adjustment$$

To estimate the RAA, one needs to first consider the explicit utility function that better represents the investors' behaviour as the utility function will have implications of the methods used to estimate RAA.

Following Bliss and Panigirtzoglou (2004), the relationship between RND: $q(St)$, RWD: p[St] and the utility function U[St] to be considered must hold:

$$\frac{p(S_T)}{q(S_T)} = \lambda \frac{U'(S_T)}{U'(S_t)}$$

The two main versed utility functions in the literature are the power utility function and the exponential utility function. Both functions represent the risk aversion coefficient with the Greek letter Gamma (ϒ); however, the type of utility function has implications in the estimation of the RAA. As the exponential utility function has constant absolute risk aversion (CARA):

$$U(x) = -e^{\alpha x}/\alpha$$

The RAA is dependent on the level of the index (St); therefore, the RAA measure must be given by ϒxSt. Conversely, the power utility function has constant relative risk aversion (CRRA):

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

Hence, the RAA measure is given directly by ϒ.

Bliss and Panigirtzoglou (2004) find evidence that for both the FTSE 100 and S&P 500 indexes, the exponential utility is able to provide a slightly better fit; however, due to the convenience of the CRRA and the fact that both methods tend not to diverge significantly, the power utility is also a valid method to apply when conducing risk transformational procedures.

Risk Aversion Estimation

To estimate the risk aversion coefficient the literature describes three main methods: Minimization of the Berkowitz L3 statistic, Maximum Likelihood method and Non-parametric methods.

Berkowitz (2001) introduced the L3 statistic as a mean of testing the predictive power of option implied densities. Bliss and Panigirtzoglou (2004) apply the measure to estimate the Gamma parameter. The authors estimate the parameter by minimizing the L3 statistic, following the rationale that if the L3 statistic represents the predictive power of a given distribution; then, by minimizing the statistic, the deviation from RND to RWD would also be minimized, being then the Gamma that most faithfully represents the investors' risk aversion. Furthermore, the authors conclude that applying this method to both CRRA and CARA (i.e. power and exponential utility functions respectively), produces robust estimators that behave relatively well even with variations of time to maturity or volatility level.

Following the same rational above, the maximization of the likelihood function when transforming RND into RWD yields the risk aversion coefficient that better represents investors' risk preferences. Liu et al (2007) estimate RND using splines, GB2 and MLN and derive a closed form solution for the risk transformation into RWD. Moreover, they conclude that RWD has higher predictive power and further compare that the two parametric methods performed better than the splines for the FTSE 100 index.

Other Methods

Falker and king (1990) propose an alternative to the methods mentioned. The authors transform the RND obtained from commodity derivatives data directly into RWD through a Beta transformation method. This method is not dependent on the utility function and produces sound estimations of the RWD (Liu et al, 2007). Liu et al (2007) apply this method to the transformation of RNDs estimated by GB2 and MLN for the FTSE 100.

Ait-Sahalia and Lo (2000) estimate the state price density through a Nadaraya-Watson Kernel assuming investors' utility the takes form of a logarithmic utility function. The authors conclude that the investors are risk averse and that the risk aversion coefficient is not constant as it increases with extreme values for the S&P 500 index.

Perignon and Villa (2002) apply a similar method to high frequency option prices on the CAC40. The authors follow Ait-Sahalia and Lo (2000) and estimate the IVF using a similar Kernel method and further derive the RWD with a Gaussian Kernel and conclude that the densities estimated are stable through time, being insensitive to the kernel chosen but sensitive to the bandwidth amplitude.

Prospect theory and rank dependent utility

Daniel Bernoulli (1954) firstly proposed the expected utility theory as a mean of describing the decision-making process of individuals under uncertainty. This theory states that rather than deciding through an expected value basis, agents' decision is driven by the utility that each situation provides them (i.e. agents are von Neumann-Morgenstern utility theorem rationale). Kahneman and Tversky (1979) on the other hand consider that the expected utility alone is insufficient to fully explain the rationale behind the agents' actions (i.e. the theory considers the outcomes to be linearly weighted and does not solve the Allais paradox[Allais,1953]). Al-Nowaihi and Dhami (2010) unfold that the agents' often attribute certainty or impossibility to events with extremely high and low probabilities, respectively. Kahneman and Tversky (1979) refer to this phenomenon as the *"ignorance at endpoints"* that is reflected in the model by the overweighting of tail events (Polkovnichenko and Zhao, 2012). Furthermore, Post et al. (2004) reports that the loss-aversion function presented in the theory reflects the agents' decision-process better than the EU theory.

Quinggin (1982) develops the Rank dependent utility theory, considering the basis of the expected utility theory while accounting for the observations posed by the prospect theory. The author introduces a rank-dependent weighting function to the cumulative probability to underweight high-probability events and overweight low-probability events. Tversky and Kahneman (1992) incorporate Quinggin's work to reflect the agents' optimism over low-probability events and the pessimism over high probability events, attributing different weighting functions to losses and gains

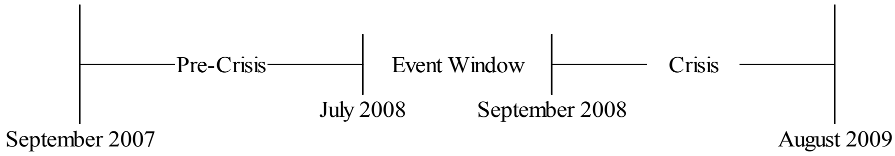
Data

To estimate the impact of the financial crisis in the risk neutral densities, as well as its influence in the coefficient of risk aversion, European call option prices (bid-ask) on the S&P 500, along with its strike prices were downloaded from the WRDS platform. The prices are relative to 3-week contracts (i.e. 15 trading days). Additionally, the dividend yield relative to the index was extracted from the same source. For the risk-free rate of return, the three-month Euro-currency interest rate for sterling pound was used as a proxy given its high liquidity.

Concerning the choice of the dates and definition of the period length, I follow Bliss and Panigirtzoglou (2004) and estimate the RNDs from 10 continuous months for each period. To choose the window of the event, I opted to use a window of three months for the event as I considered the subprime mortgage crisis to be a succession of events rather than a single isolated event.

I opted not to use the bankruptcy of the bear sterns (January 2007) as the beginning of the event window because the Fed bailed out the bank. After the bail out, the US equity market rebounded and by May, both the Dow Jones and the S&P 500 had recovered from the losses imposed. By then, besides the weak GDP growth of 2007's last quarter and FED's announcement on shortage of liquidity in banks, it would be difficult to guess that such a crisis was approaching.

In that sense, I consider the bankruptcy of the Fannie Mae and Freddie Mac (July 2008) as the beginning of the event window. The reasoning is that most investors only realised that a financial crisis was coming by then (when the Mortgage backed securities started to fail its payments). To not include the event in my estimations, I set the pre-crisis period to end in the last expiration date prior to July 2008. The pre-crisis period ranges from September 6, 2007 (first information date) to June 21, 2008 (last expiration date). Furthermore, I consider the end of the event window to be the bankruptcy of the Lehman Brothers, the bailout of AIG and the blocking of the bailout bill by the US senate, all happening in September 2008. Consequently, the crisis estimation period is set to start 15 trading days before the first expiration date after the senate first vote. The crisis period is then set from November 6, 2008 (first information date) to August 22, 2009.



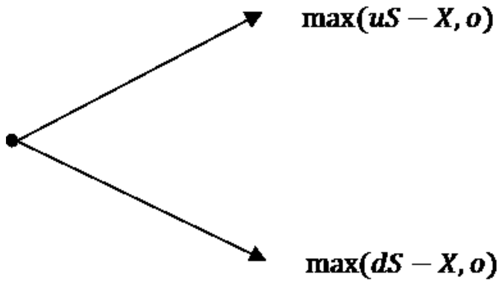
Methodology

Risk neutral Density Estimation

Consider a Binary option[1] paying \$1 at maturity with probability p and \$0 with probability (1- p). Following the Arbitrage price theory, the price of such asset is the present value of the expected payoffs at maturity T (Taylor, 2005). The theoretical price would then be (being r the risk-free rate of return and q the underlying's dividend yield):

$$price(x) = e^{-(r-q)T}[p * \$1 + (1-p) * \$0] \qquad (1)$$

If we now consider a European call option, we obtain a payoff scheme such as:



Following equation 1, the theoretical price of the call option is:

$$c(X) = e^{-(r-q)T}[p * \max(uS - X, 0) + (1-p) * \max(dS - X, 0)] \qquad (2)$$

In both scenarios, *"there is a risk-neutral probability that prevents arbitrage profits"* (Hull 2000, Chapter 9). This probability is the risk-neutral probability and it differs from

[1] Asset that pays a pre-determined payoff at maturity T contingent on the underlying event happening or not.

the real-world probability whenever investors are risk averse (Taylor, 2005) due to the presence of risk premium.

In this simplistic context, we can represent the theoretical price of the European call as the risk-neutral probability-weighted expected payoff (E_Q) considering only the positive payoff (Taylor, 2005):

$$c(X) = e^{-(r-q)T} * E_Q[S_t - X] \tag{3}$$

Following this reasoning, if we consider a set of strike-prices large enough so that it is representative of a complete market, for x ≥ 0 :

$$c(X) = e^{-(r-q)T} * \int_{x}^{\infty} (x - X) f_Q(x) dx \tag{4}$$

European call option prices (or European put option prices converted to call option prices by the put-call parity) allow to directly estimate the implied risk-neutral density, above represented by $f_Q(x)$, by matching the observed market price with the present value of the risk-neutral probability weighted expected payoff for the set of call option prices available. For $f_Q(x)$ to be considered a density function the following conditions must be verified:

$$f_Q(x) \geq 0 \; and \int_{0}^{\infty} f_Q(x) dx = 1 \tag{5}$$

For each strike price (X) and for all the methods considered below, the market price for each option was considered to be the mean value between the best bid and the best offer:

$$Market\ Value_X = \frac{Best\ bid_X + Best\ offer_X}{2} \tag{6}$$

The Mixture of Lognormal distributions, GB2 and the log-normal polynomial (expansion) were used in this paper to extract the above-mentioned implied risk-neutral densities from option prices.

Mixture of Log-normal distributions

This methodology assumes that the implied RND is best described by the weighted combination of two distinct log-normal distributions. The density function for the RND obtained with this method is thus:

$$f_Q(x) = p\psi(x|S_1, \sigma_1, T) + (1-p)\psi(x|S_2, \sigma_2, T) \quad (7)$$

Where Si is the index value for each of the lognormal distributions, σi the annualized volatility of each distribution and p the weight attributed to each of the distributions. Each lognormal distribution assumes the following form:

$$\psi(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left(-\frac{1}{2}\left[\frac{\log(x) - \log(S) + (-r + q + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right]^2\right) \quad (8)$$

Given0 $< p < 1$ and the estimated parameter vector $\theta = \{Si, \sigma i, p\}$, f_Q is risk neutral if:

$$S = p * S1 + (1-p) * S2 \quad (9)$$

This constraint reduces to four the number of parameters to estimate.

Under this methodology, the theoretical Black-Scholes call price is logically given by a mixture of two black-Scholes call prices obtained from each of the distributions, under the assumptions that St follows a lognormal distribution:

$$c(X|\theta, r, T) = pc_{BS}(S_1, T, X, , r, q, \sigma_1) + (1-p)c_{BS}(S_2, T, X, , r, q, \sigma_2) \quad (10)$$

To estimate the parameters, it is straightforward to minimize G(X), defined as the sum of the squared deviations between the observed market price and the obtained BS theoretical call price for the set of *N* strike prices available:

$$G(\theta) = \sum_{i=1}^{N} [c(X_i) - C(X_i|\theta)]^2 \quad (11)$$

The minimization is subject to the following constraints:

$$S_1 > 0\,, S_2 > 0\,, \sigma_1 > 0\,, \sigma_2 > 0$$

This method to extract RND is especially useful to model events with two distinct outcomes such as a presidential re-election or a fed announcement on possible interest rate change (Taylor, 2005).

For this methodology, the raw moments can be computed using the moment generator function provided by Taylor (2005) for the n^{th}:

$$E[S_T^n] = pS_1^n \exp\left(\frac{1}{2}(n^2 - n){\sigma_1}^2 T\right) + (1-p)S_2^n \exp\left(\frac{1}{2}(n^2 - n){\sigma_2}^2 T\right) \quad (12)$$

The raw moments were then converted to central moments μ_n, to compute variance, skewness and kurtosis:

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} {\mu'}_k {\mu'^{n-k}}_n \quad (13)$$

Generalized Beta Distribution of the second kind

Following McDonalds and Bookstaber (1987) methodology achieves greater freedom regarding the shape of the distribution by allowing for a broad range of combinations for the first four moments of the distribution, being characterized by the parameter vector $\theta = \{a, b, p, q\}$.

The parameters' interpretation is broadly discussed in the literature being the general consensus is that the parameter b is a scale parameter while aq determines the number of finite moments. Nevertheless, the individual interpretation of the three parameters is unclear (Taylor, 2005).

The density function for the GB2 is defined as follows, given $x > 0$:

$$f_{GB2}(x|a, b, p, q) = \frac{a}{b^{ap} B(p, q)} \frac{x^{ap-1}}{\left(1 + \left(\frac{x}{b}\right)^a\right)^{p+q}} \quad (14)$$

Being the Beta function, defined as a combination of gamma functions:

$$B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q) \tag{15}$$

For the Beta distribution, the moment generator function is, for the n^{th} moment of the distribution (with $n < aq$):

$$E[S_T^n] = \frac{b^n B(p+\frac{n}{a}, q-\frac{n}{a})}{B(p,q)} \tag{16}$$

The literature stresses that moments of higher order than aq do not exist and that the kurtosis is infinite whenever $aq < 4$ (Taylor, 2005). Additionally, solving for the first moment of the distribution (i.e. n=1), while assuming $aq > 1$ we obtain the risk neutrality constraint:

$$S = \frac{e^{-(r-d)T} bB\left(p+\frac{1}{a}, q-1/a\right)}{B(p,q)} \tag{17}$$

This is especially useful when estimating the parameters as we can now derive the b parameter from the other parameters:

$$b = \frac{Se^{(r-d)T}B(p,q)}{B(p+\frac{1}{a}, q-\frac{1}{a})} \tag{18}$$

Liu et al defines the closed form solution for computing the call price using GB2:

$$\begin{aligned} c(X|r,T) = Se^{-rT}\left[1 - G_\beta\left(z(X,a,b)|p+\frac{1}{a}, q-1/a\right)\right] \\ - XSe^{-dT}\left[1 - G_\beta(z(X,a,b)|p,q)\right] \end{aligned} \tag{19}$$

Being that the call price depends on c.d.f of the GB2 distribution, this function can be evaluated using the c.d.f. of the Beta distribution (incomplete beta function):

$$F_{GB2}(x|a,b,p,q) = F_{GB2}\left(\left(\frac{x}{b}\right)^a \middle| 1,1,p,q\right) = F_\beta(u(x,a,b)|p,q) \tag{20}$$

With:

$$u(x,a,b) = \frac{\left(\frac{x}{b}\right)^a}{1+(x/b)^a} \tag{21}$$

Being the call price function then:

$$\begin{aligned} c(X) \ &= Se^{-rT}\left[1 - F_\beta\left(u(x,a,b)|p + \frac{1}{a}, q - 1/a\right)\right] \\ &- XSe^{-dT}\left[1 - F_\beta(u(x,a,b)|p,q)\right] \end{aligned} \tag{22}$$

The estimation of the parameters is once again done by minimizing pricing error function G(θ) function by estimating the vector parameter $\theta' = \{a, p, q\}$ while obtaining the parameter b through equation 16:

$$G(\theta) = \sum_{i=1}^{N} [c(X_i) - C(X_i|\theta)]^2 \tag{23}$$

It is interesting to mention that the GB2 as a distribution belonging to the generalized distribution family, having many well-known distributions as special cases. As example, when a tends to zero, the GB2 tends to a lognormal distribution (ceteris paribus). When q tends to infinitely large values, the distribution tends to a gamma distribution (Appendix 1 presents further information about the limiting cases of the GB2).

Lognormal polynomial expansion

Following Madan and Milne (1994), it is assumed that standardized returns' (Z) RND is constituted by a standard normal density multiplied by a polynomial function:

$$f_z(z) = \phi(z) \sum_{j=0}^{\infty} b_j H_j(Z) \tag{24}$$

Being bj constants and Hj the normalized orthogonal Hermite polynomials:

$$H_0(z) = 1, \qquad H_1(z) = z, \qquad H_2(z) = \frac{1}{\sqrt{2!}}(z^2 - 1), \qquad H_3(z) = \frac{1}{\sqrt{3!}}(z^3 - 3z),$$
$$H_4(z) = \frac{1}{\sqrt{4!}}(z^4 - 6z^2 + 3), \quad \dots$$

and $\phi(Z)$ the normal distribution for the standardized returns (Z) that are defined as:

$$Z = \frac{\ln\left(\frac{S_T}{S}\right) - (\mu T - \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}} \tag{25}$$

Where St corresponds to the forward price at maturity date while S stands for the spot price of the index (i.e. at information date). Moreover, two parameters μ and σ are defined assuming that the log returns' RND has finite variance ($\sigma^2 T$) and mean defined by $\mu T - 0.5\sigma^2 T$. Finally, under RND assumptions the forward price F_t equals the future spot price at maturity date S_t.

Following the authors reasoning, the density of prices follows a lognormal density multiplied by a polynomial function of the log prices. The relationship between the standardized returns' density and the price density is therefore:

$$f_{S_T(x)} = \frac{1}{x\sigma\sqrt{T}} f_z(z) \tag{26}$$

Furthermore, the orthogonal property of the Hermite polynomials implies that:

$$\int_{-\infty}^{\infty} h_i(z) h_j(z) \phi(Z) dz = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \tag{27}$$

The choice of orthogonal polynomials is especially interesting as it allows for equation 22 to be solved in order to bj in a simpler way:

$$\int_{-\infty}^{\infty} H_j(z) f_z(z) dz = bj \tag{28}$$

Equation 26 is of special importance as it allows for the drawing the essential constraints in the estimation of the model:

1)$B_0 = 1$ for the f(z) to be considered a density function:

$$\int_{-\infty}^{\infty} f(z)dz = 1$$

2)$B_1 = 0$ for the mean to be zero:

$$z\int_{-\infty}^{\infty} zf(z)dz = 0$$

3) $B_2 = 0$ for the variance to be one:

$$\int_{-\infty}^{\infty} z^2 f(z)dz - \left(\int_{-\infty}^{\infty} zf(z)dz\right)^2 = 1$$

The implementation of this method requires $f_z(z)/\phi(z)$ to be a polynomial of finite order. Madan and Milne (1994) considers polynomials of the forth order, implying $b_j = 0$ for $j \geq 5$. Along with the restriction considered above, the parameter vector to be estimated is then $\theta = \{\mu, \sigma, b_3, b_4\}$.

Considering the standardized return of the underlying security to be:

$$\frac{S_T}{S} = e^{\mu T - \frac{1}{2}\sigma^2 T + Z\sigma\sqrt{T}} \tag{29}$$

The option's payoff is therefore:

$$\max(S_t - X; 0) = \max(Se^{\mu T - \frac{1}{2}\sigma^2 T + Z\sigma\sqrt{T}} - X; 0) \tag{30}$$

For the lognormal-polynomial, the positive call payoff can be written as:

$$\sum_{K=0}^{\infty} a_K(S, T, X, \mu, \sigma)h_k(Z) \tag{31}$$

Considering equation 3 and 29, the call price for the lognormal-polynomials methodology is then given by:

$$c(X) = e^{-(r-q)T} \sum_{j=0}^{\infty} a_j(S,T,X,\mu,\sigma) b_j \tag{32}$$

Being a_Ka function that does not depend on Z. By applying the orthogonal property and integrating over $-\infty < Z < \infty$, a_K is defined as:

$$a_K(S,T,X,\mu,\sigma) = \int_{-\infty}^{\infty} \left[S e^{\mu T - \frac{1}{2}\sigma^2 T + Z\sigma\sqrt{T}} - X\right]^+ h_K(Z)\phi(Z)dz \tag{33}$$

Considering the abovementioned constraints: $b_0 = 1, b_1 = 0, b_2 = 0$ and $b_j = 0$ for $j \geq 5$, there is only need to compute a_0, a_3 and a_4:

$$a_0(S,T,X,\mu,\sigma) = Se^{\mu T} N(D_1) - XN(D_1 - \sigma\sqrt{T})$$

$$a_3(S,T,X,\mu,\sigma) = \frac{\beta}{\sqrt{6}} Se^{\mu T}[\beta^2 N(D_1) + (2\beta - D_1)\phi(D_1)]$$

$$a_4(S,T,X,\mu,\sigma) = \frac{\beta}{\sqrt{24}} Se^{\mu T}[\beta^3 N(D_1) + \left(3\beta^2 - 3\beta D_1 + {D_1}^2 - 1\right)\phi(D_1)]$$

With:

$$\beta = \sigma\sqrt{T} \qquad , \qquad D_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(\mu + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Finally, to estimate the parameter vector $\theta = \{\mu, \sigma, b_3, b_4\}$, the method is once again the minimization of the mispricing error as applied in the previous method, subject to the risk neutrality constraint:

$$1 + \frac{\beta^3 b_3}{\sqrt{6}} + \frac{\beta^4 b_4}{\sqrt{24}} = e^{-\mu T} \tag{34}$$

Risk Aversion estimation from utility function

Assuming a risk averse world, RND must be transformed into real world densities (RWD) to reflect the risk aversion level of the market participants.

In this study, the utility function assumed to represent investors' behavior was the power utility:

$$u(w) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} & ,\gamma \neq 1 \\ \ln(w) & ,\gamma = 1 \end{cases} \tag{35}$$

This utility function has constant relative risk aversion (CRRA) given by the parameter gamma γ. Following Liu et al (2007), the coefficient is estimated through maximum likelihood. Being $\tilde{f}$ the real-world density at time t_i; θ_i, the known RND parameter vector and θ^*, the real-world vector of parameters to be estimated by jointly maximizing the log-likelihood function:

$$\log\left(L\left(S_{T,1}, S_{T,2}, \ldots, S_{T,n} \middle| \theta *\right)\right) = \sum_{i=1}^{n} \log(\tilde{f}(S_{T,i}|\theta_i, \theta^*) \tag{36}$$

Applying the real-world transformation to the mixture of lognormal distributions requires the real-world vector of parameters $\theta' = \{S_i{}^*, \sigma_i{}^*, p^*\}$ to be estimated. Considering the following relations:

$$S_i{}^* = S_i e^{\gamma \sigma_i{}^2 T}$$

$$p^* = \frac{1}{1 + \frac{1-p}{p}(\frac{S_2}{S_1})^{\gamma} e^{0.5(\gamma^2-\gamma)(\sigma_2{}^2-\sigma_1{}^2)T}}$$

The likelihood function is maximized by changing the CRRA coefficient ϒ.

Risk aversion estimation from rank-dependent utility function

Unlike the expected utility theory, the rank dependent expected utility does not attribute linear weights to the probabilities. Quiggin (1982) reports that agents overweight the probability of unlikely extreme outcomes. Moreover, Kahneman and Tversky (1979) report that agents' decisions are based on gains and losses rather than in wealth; therefore, the model must reflect such differences. π is the weighting function that alters the weight of the probabilities. This function is continuous, differentiable, non-decreasing and has no weighting effect on probabilities of 1 or zero. Furthermore, In a rank-dependent utility model, the following characteristic holds:

$$\pi^{-}(p) = 1 - \pi^{+}(1-p) = \pi(p) \tag{37}$$

The utility is given by:

$$U = \int_{-\infty}^{\infty} u(w) d\pi(P(w)) \tag{38}$$

Considering $q(S)$ to be the RND, $u'(S)$ to be the utility function representative of investors behavior and P(S) the real-world CDF:

$$q(S) = \frac{u'(S)}{E_{\pi P}\big(u'(S)\big)} \pi'(P(S))P'(S) \tag{39}$$

This relation can be re-written as:

$$P(X) = \pi^{-}\left(\frac{\int_0^x \frac{q(S)}{u'(S)}}{\int_0^{+\infty} \frac{q(S)}{u'(S)} dS} \right) \tag{40}$$

As mentioned above, to truly represent agents' behavior, the weighting function π needs to overweight the probability of unlikely extreme events (i.e. this function has then an inverse-S shape). Prelec (1998) proposes a functional form allowing to control for the relative weight of the concave and convex parts of the curve:

$$\pi_2(P) = e^{-(-\beta lnP)^{\alpha}} \tag{41}$$

The inverse function can be solved to y (RND's CDF):

$$\pi_2{}^{-1}(y) = e^{-\frac{(-\ln y)^{1/\alpha}}{\beta}} \tag{42}$$

Applying the RDEU to the mixture of lognormal distributions, the C.D.F. is given by:

$$y' = 0.5\left(1 + erf\left(\frac{\ln(S_T) - u}{\sqrt{2}\sigma\sqrt{T}}\right)\right) \tag{43}$$

With

$$\text{erf}(x) = \frac{1}{\sqrt{x}} \int_{-x}^{x} e^{-t^2} dt$$

$$u = \ln(S^*) + (r\text{-}d)T - 0.5\sigma^2 T,$$

$$S^* = Se^{\gamma\sigma^2 T}.$$

Finally, the parameter vector $\theta' = \{\gamma, \alpha, \beta\}$ are estimated by maximizing the log-likelihood function.

Empirical Findings

Risk Neutral Densities

On account of consistency, the number of non-overlapping months, time to expiration of contracts as well as the methods applied to each data-set were kept constant throughout the analysis.

To estimate the risk neutral densities for both the pre-crisis and the crisis periods, a total of 20 non-overlapping months of European calls on the S&P 500 index were analysed under three different parametric methodologies: Mixture of lognormal distributions, GB2 and Lognormal-polynomials. Furthermore, the estimation of the risk neutral densities was achieved by minimizing the sum of squared deviations of the theoretical price with respect to the observed market price, $G(\theta)$.

Each month's risk neutral density (f_Q) was estimated considering options with a time to expiration of 15 trading days. The expiration dates are standardized for the derivatives market, being each month's expiration date, the Saturday following the third Friday of the month. A total of 20 prediction dates (i.e. information dates) were considered.

Firstly, for each data set the quality of each model regarding its pricing error is accessed and the moments of the distribution obtained are described to provide a general introduction to the results obtained for each data set. Secondly, the two data sets are compared relatively to the general state of the index to set the basis for the final section where I analyse the distributions obtained right before and after the event period.

Pre-crisis period

The period considered in this data set ranges from September 22, 2007 (i.e. first expiration date) to June 21, 2008 (i.e. last expiration date) encompassing 10 prediction dates in total.

The ability of each model to match theoretical price with the observed data was contrasting. The Mixture of Lognormal distributions was able to achieve the lowest sum of squared pricing errors, $G(\theta)$, followed by the Lognormal-Polynomial and lastly by the GB2. Table 1 summarizes the findings.

Table 1
This table summarizes the quality assessment regarding the chosen models. Total $G(\theta)$ represents the average of the aggregated mispricing computed for each month, Average represents the average mispricing per option, Minimum and Maximum represent the average minimum and maximum $G(\theta)$ obtained for each month and standard deviation represents the average standard deviation of the $G(\theta)$ given by each model's estimates. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The time frame considered was from September 22, 2007 until June 21, 2008.

Average G(θ)	MLN	GB2	Polynomial
Total	151,12	474,87	281,34
Average	2.03	4.39	3.05
Minimum	0,00	0,00	0,00
Maximum	5,92	12,34	8.07
Standard Deviation	2,05	0,33	2,91

Besides considering the pricing error of each methodology, it is crucial to consider another aspect that is particular to the lognormal-polynomial methodology. This method does not guarantee compliance with the non-negativity constraint (i.e. a density function cannot assume negative values). Following Jondeau and Rockinger (2001) recommendations, skewness was restricted to $\pm 1{,}5$ while kurtosis was allowed to fluctuate between 3 and 7. This correction does not guarantee compliance with the constraint following that in some months, negative densities might still be observed, thus inference derived from those months would not be valid.

From the ten risk neutral distributions, nine initially presented negative values. The abovementioned procedure was able to correct all the nine months' densities; thus, inference is valid even though the solutions obtained were not the yielding the lowest $G(\theta)$.

From Table 1, the method with best performance was the mixture of lognormal distributions, achieving an average (aggregated) mispricing of 151,12 per month's estimate across all ten months. This measure consists in the sum of mispricing across the set of options prices, considering for its average the data from the 10 months. Furthermore, this method also achieved the lowest average $G(\theta)$, indicating that on average, the mispricing per option price was the lowest amongst the three models. Finally, while the GB2 method achieved the lowest standard deviation, it also delivered the highest average $G(\theta)$ as well as highest average maximum $G(\theta)$.

Regarding the moments of the distribution, the moment generator function was used for the computation of the first four raw moments, except in the lognormal-polynomial method as this method offers the closed form solution for the four central moments. For the mixture of lognormal distributions and for the GB2, the raw moments were then converted to central moments according to the methodology. Table 2 summarizes the findings.

Table 2

This Table reports the average values for first four central moments of the distribution: Mean, Standard Deviation, Skewness and Kurtosis. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The time frame considered was from September 22, 2007 until June 21, 2008.

Moments	MLN	GB2	Polynomial
Mean	1 421,62	1 427,38	1 426,33
Standard Deviation	54,27	54,99	58,55
Skewness	-0,44	-0,62	0,72
Kurtosis	3,00	4,17	4,17

From table 2, the mean ranges between \$1 421,62 (MLN) and \$1 427,38 (Polynomial) and represents the average of the risk-neutral expectations for the index value at the maturity date. The standard deviation reflects the average dispersion of the expectations around the mean (referent to the 15 trading-days period), ranging from \$54,27 (MLN) and \$58,55 (Polynomial). The skewness is negative for both Mixture of lognormal distributions and GB2 and it is positive for the Lognormal-Polynomial method. Finally, the Kurtosis was positive in the three methods employed being 4,13 for both the GB2 and Polynomial while the MLN kurtosis was 3,00.

Crisis period

The period considered in this data set ranges from November 22nd, 2008 (i.e. first expiration date) to September 19th, 2009 (i.e. last expiration date) encompassing 10 prediction dates in total.

Following the same reasoning as above, the relative estimation quality for the three methodologies remained constant. In this data set the Mixture of Lognormal distributions was the methodology that reported the lowest pricing error $G(\theta)$, followed by the Lognormal-polynomial and lastly by the GB2. Table 3 reports the findings.

Table 3

This table summarizes the quality assessment regarding the chosen models. Total $G(\theta)$ represents the average of the aggregated mispricing computed for each month, Average represents the average mispricing per option, Minimum and Maximum represent the average minimum and maximum $G(\theta)$ obtained for each month and standard deviation represents the average standard deviation of the $G(\theta)$ given by each model's estimates. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The time frame considered was from November 22nd, 2008 until September 19th, 2009.

Average G(θ)	MLN	GB2	Polynomial
Total	76,09	163,30	92,57
Average	0,81	1,03	0,97
Minimum	0,00	0,00	0,00
Maximum	2,91	4,78	2,67
Standard Deviation	0,93	1,34	0,90

Analogously to the pre-crisis data set, the lognormal-polynomial methodology also produced negative densities for the crisis data set. From the ten densities estimated in this data set, five presented negative densities. Applying the Jondeau and Rockinger (2001) restriction, all the five densities were corrected, being the inference drawn from the polynomial methodology valid.

From Table 3, the method with best performance was the mixture of lognormal distributions, achieving an average (aggregated) mispricing of 76,09 per month's estimate across eleven months. Furthermore, this method achieved, once more, the lowest average $G(\theta)$, indicating that on average, the mispricing per option price was the lowest amongst the three models. Conversely to the prior data set, the Lognormal-polynomial achieved the lowest standard deviation. Even though it also achieved lower average maximum

$G(\theta)$, the higher average $G(\theta)$ per option indicates that this methodology performed consistently below the mixture of lognormal distribution method.

Concerning the moments of the estimated distribution, the same procedure was followed regarding the raw and central moments. Table 4 presents the average values for the first four central moments given by each methodology.

Table 4

This Table reports the average values for first four central moments of the distribution: Mean, Standard Deviation, Skewness and Kurtosis. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The time frame considered was from November 22nd, 2008 until September 19th, 2009. Mean and standard deviation are in $ amount.

Moments	MLN	GB2	Polynomial
Mean	902,15	896,26	887,73
Standard Deviation	69,44	64,66	68,22
Skewness	-0,92	-0,51	0,71
Kurtosis	4.86	3,70	4,21

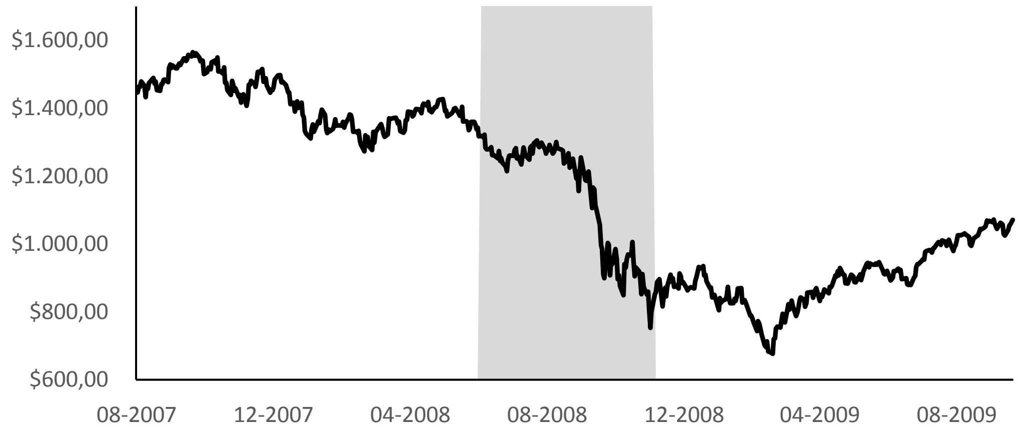
From table 4, the mean ranges between $902,15 (MLN) and $887,73 (Polynomial), representing the average of the risk-neutral expectations for the index value at the maturity date. The standard deviation reflects the average dispersion of the expectations around the mean (referent to the 15 trading-days period), ranging from $69,44 (MLN) and $64,66 (GB2). The skewness is negative for both Mixture of lognormal distributions and GB2 and it is positive for the Lognormal-Polynomial method. Finally, the Kurtosis was positive in the three methods employed, ranging from 4,86 (MLN) and 3,70 (GB2).

Comparison between periods

The methodologies employed in the estimation of the risk neutral densities were able to successfully fit the observed market prices to the underlying models while respecting the imposed restriction. From table 1 and 3, the mixture of lognormal distributions outperformed the other methods across both samples by achieving lower agglomerated pricing error and lower average mispricing per option evaluated. Whenever other methods outperformed in the remaining measures, the MLN followed closely the top performer. Given the prior considerations, the following comparison between both periods will be carried out using data respective to the MLN method.

Figure 1

This figure depicts the evolution of the price for the S&P 500 composite index. The black line represents the index level while the grey area represents the event window. The white to the right of the event window depicts the pre-crisis period while the white area to the right of the event window represents the crisis period. The time frame ranged from September 22, 2007 to September 19th, 2009. The event window ranges from June 22th, 2008 to November 22nd, 2008



From tables 2 and 4, the mean value for the risk neutral expectations suffered an accentuated decrease, from $1 421,62 to $902,15. This drop is consistent with the price evolution for the S&P 500 index depicted in figure 1. In the beginning of the event window, the index level was $1 478,55 while at the end was $851,81.

The average standard deviation derived from the implied distribution increased from $54,27 to $69,44; thus, on average, the dispersion around the mean of the predictions increased, which is consistent with the times of uncertainty associated with the crisis period. Furthermore, the skewness became more negative, from -0,44 to -0,92. This signals that on average, during the crisis period, the distribution had longer left tails when compared to the pre-crisis period. Negative skewness is associated with higher probability of price drops; hence, the probability of losses increased in the crisis period, as expected. Finally, on average, the kurtosis increased from 3,00 to 4,86. This is a relevant change and is associated with higher density in the tails of the distribution (i.e. fat tails). Again, the probability of extreme events (i.e. accentuated price movements) is higher in the crisis period, which is consistent with the higher uncertainty that characterizes crisis periods.

RND comparison

To better understand the impact of the crisis in the risk neutral distributions, let us consider the last estimation period before the event window and the first estimation period after the event window. For this analysis, the output derived from the three methodologies will be compared and further considerations will be drawn from their results.

Figure 2

This figure presents the implied risk neutral distributions estimated for the expiration date of June 21st, 2008 (pre-crisis), using the three methodologies. The black line represents the RND estimated using the mixture of lognormal distributions, the dashed line represents the RND obtained through the GB2 and the dotted line represents the RND estimated through the Lognormal-Polynomial.

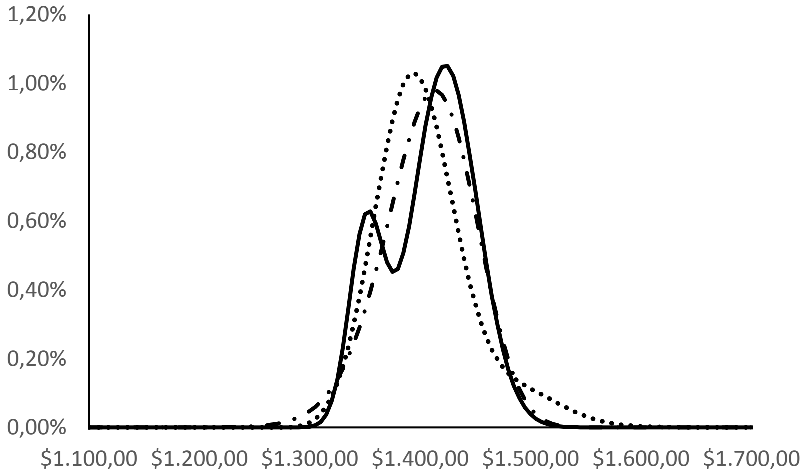
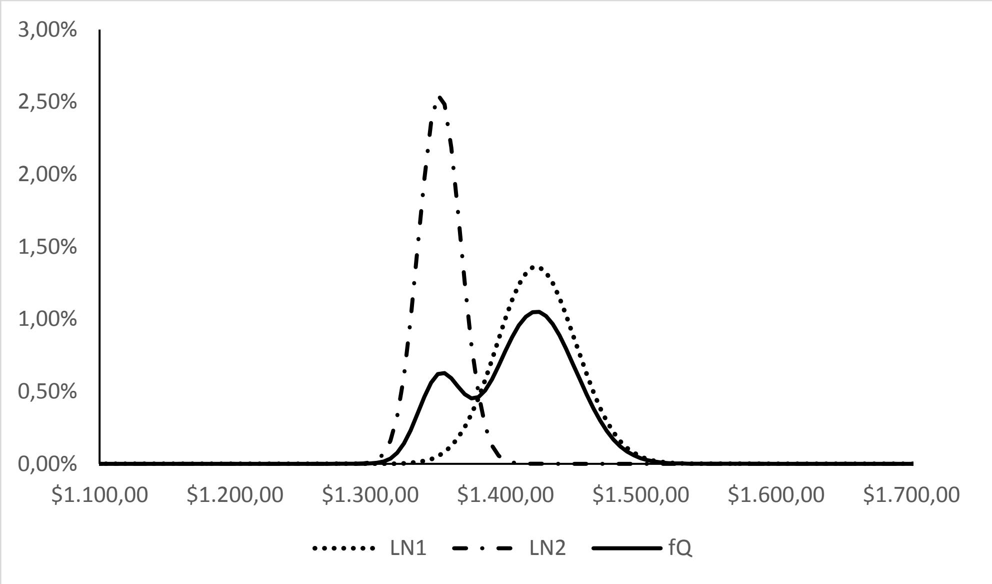


Figure 2 depicts the densities obtained for the last expiration date, June 21st, 2008. The densities obtained differ in shape and in characteristics, especially the MLN density. It differs from the remaining distributions by reporting an accentuated bimodal distribution. This shape is due to the fact that the parameter p assumes the value of 77% and the two lognormal distributions that are part of the final distributions have distinct shapes.

Considering two possible states: the past events, such as the announcement of an almost null growth of the US GDP during 2007 (in January 2008) or the bankruptcy and bailout of the Bear Sterns Bank (March 2007), leading to a serious financial crisis or not. Theoretically, such different states would imply distinct densities. Arguably, this might be the cause for such pronounced increase of density in the left tail (i.e. the two densities are well captured by the mixture of lognormal distributions), whose constitution is decomposed in detail in figure 3.

Figure 3

This figure presents the implied risk neutral distributions estimated for the expiration date of June 21st, 2008 (pre-crisis), using the mixture of lognormal distributions (black line) and its components: the dashed line represents the First lognormal distribution and the dotted line represents the represents the second lognormal distribution. The probability (p) associated with the first lognormal distribution is 77%.



The probability of 27% referent to the second lognormal distribution may pose that the participants in the market were already considering the possibility of a sharp price decrease in the near future for the S&P 500 Index. The fact is that the index ended up even below expectations (i.e. closing price at expiration date was $1 317,93).

From table 5, it is difficult to confirm the hypothesis posed above as the moments of the distribution are not consistent with such views. For all methods, the mean ranged from $1 404,05 (MLN) and $1 407,26 (GB2). For the MLN and GB2 methods, the skewness increased relatively to the average presented in table 2 while the kurtosis decreased. The aggregated effect would be thinner tails and shorter left tail for both densities. However, the fact that the MLN density is bi-modal, might justify the increase of skewness and decrease of kurtosis for the MLN density. Regarding the density extracted from the polynomial methodology, the values were in line with the average presented in table 2.

Table 5

This Table reports the average values for first four central moments of the distribution: Mean, Standard Deviation, Skewness and Kurtosis. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The results are referent to the density estimated for the expiration date June 21st, 2008.

Moments	MLN	GB2	Polynomial
Mean	1 404,05	1 407,26	1 405,02
Standard Deviation	40,31	41,40	44,89
Skewness	-0,17	-0,30	0,71
Kurtosis	2,32	3,19	3,90

Figure 4 depicts the densities estimated for the first expiration date in the crisis period, November 22nd, 2008. Both the MLN and GB2 densities indicate the presence of negative skewness, with the presence of longer left tails. Table 6 confirms this fact by reporting a relevant decrease in skewness when compared to the values presented in table 4 relative to the pre-crisis period. Furthermore, the Lognormal-polynomial continues to produce positively skewed densities, while reporting, kurtosis levels in line with MLN and GB2 densities, that are above the levels reported in pre-crisis densities. Moreover, the standard deviation of the risk-neutral expectation increased drastically.

Altogether, the moments are complacent with a crisis period as the mean reflected the decrease observed in the index level, the standard deviation indicates high uncertainty regarding the future state of the index, the negative skewness indicates longer left tails and the higher kurtosis indicates fatter tails; hence, higher probability of extreme negative events.

Figure 4

This figure presents the implied risk neutral distributions estimated for the expiration date of November 22nd, 2008 (crisis period), using the three methodologies. The black line represents the RND estimated using the mixture of lognormal distributions, the dashed line represents the RND obtained through the GB2 and the dotted line represents the RND estimated through the Lognormal-Polynomial.

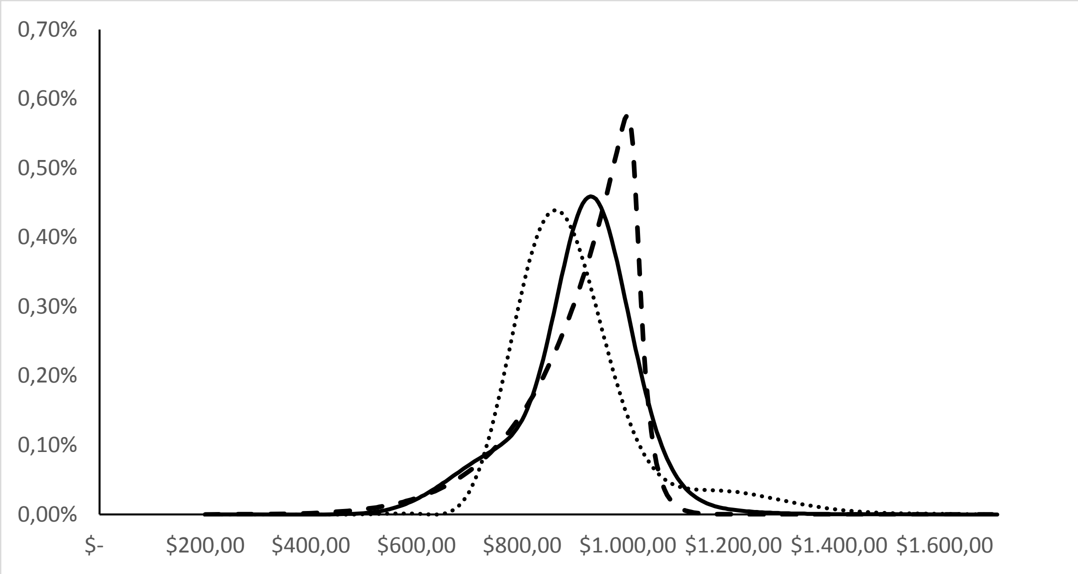


Table 6

This Table reports the average values for first four central moments of the distribution: Mean, Standard Deviation, Skewness and Kurtosis. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial). The results are referent to the density estimated for the expiration date June November 22nd, 2008.

Moments	MLN	GB2	Polynomial
Mean	904,88	906,45	904,90
Standard Deviation	110,72	111,54	118,65
Skewness	-0,46	-1,28	0,95
Kurtosis	4,07	4,91	4,60

From table 7, the percentiles are coherent with the findings discussed above. The larger relative drop in the percentiles between the two periods was referent to the 5% percentile in all methodologies employed, followed by the other percentiles in increasing order. This indicates that for the 3 methods, the left tail was more extended than any of the remaining parts of the density. Moreover, the negative skewness is depicted for MLN and GB2 (in both periods) by the larger difference from the percentile 50% towards percentile 25%

when compared to the difference towards the 75% percentile. Conversely, for the Polynomial, the opposite was true as difference relative to the right tail if larger.

Table 7

This Table reports the percentiles for the risk neutral expectations. Results are reported for three parametric models: Mixture of Lognormal distributions (MLN), Generalized Beta Distribution (GB2) and Lognormal-Polynomial (Polynomial) and are referent to the density estimated for the expiration dates June 21st, 2008 (pre-crisis) and June November 22nd, 2008 (Crisis). 5%, 25%, 75% and 100% Percentiles are reported along with the interquartile range. Comparison reports the percentage change between the pre-crisis period and the crisis period.

Pre-Crisis	MLN	GB2	Polynomial
5%	1 193,62	1 168,46	1 204,05
25%	1 233,20	1 245,74	1 239,76
50%	1 287,59	1 280,70	1 256,75
75%	1 308,47	1 302,14	1 293,98
100%	1 900,00	1 900,00	1 900,00
75%-25%	75,26	56,40	54,22
Crisis			
5%	716,05	790,89	758,33
25%	843,45	888,50	825,78
50%	921,72	932,54	880,97
75%	983,21	956,12	953,89
100%	1 700,00	1 700,00	1 700,00
75%-25%	139,75	67,63	128,11
Comparison			
5%	-40%	-32%	-37%
25%	-32%	-29%	-33%
50%	-28%	-27%	-30%
75%	-25%	-27%	-26%
100%	-11%	-11%	-11%
75%-25%	86%	20%	136%

Concluding, the estimated risk neutral densities are in line with expectations and with what the literature suggests. Figlewski (2008) finds negative skewed distributions for a period prior to the subprime crisis. Furthermore, (Dennis and Mayhew, 2002) mention that during financial crisis, skewness tends to become more negative, compared to expansion periods. Sheikh and Qiao (2010) reports that the equity returns tend to present a leptokurtic shape (i.e. kurtosis assuming values above 3) while the excess kurtosis tends to become more accentuated during financial turmoil periods. Finally, Schwert (2011) finds that standard deviation increased during the financial crisis.

Risk Aversion estimation

Following the findings presented in tables 1 and 3, the mixture of lognormal densities performed better regarding the minimization of the pricing errors; hence this section will base its estimate for the risk aversion parameter on the implied RND obtained through the mixture of lognormal distributions.

To estimate the risk aversion coefficient the power utility function was assumed as representative of the investors' preferences. Moreover, two methodologies were employed: Expected utility and rank dependent expected utility. Table 8 reports the coefficients estimated for each of the methodologies for the pre-crisis period.

Table 8

This Table reports the coefficients for the constant relative risk aversion (CRRA) along with the sum of the log-likelihood function. The coefficients were estimated from the ten Mixture of Lognormal distributions (MLN) risk neutral densities estimated beforehand. The time frame considered ranged from September 22, 2007 (First expiration date) to June 21th, 2008 (Last expiration date).

Pre-Crisis	Expected Utility	Rank-Dependent
Υ	2,81	4,41
Σ Log-Likelihood	-57,37	-55,48

Positive risk aversion coefficients are complacent with the risk premium puzzle. This puzzle was firstly described by Mehra and Prescott (1985) referring to the abnormally high equity premium over bond returns (i.e. 6,99% over the past 100 years). Moreover, the authors pose that the given equity premium would imply abnormally high (positive) risk aversion parameters. So far, the literature is yet to propose a solution for this intriguing puzzle. From table 8, The CRRA coefficient obtained through EU and RDEU estimations are both positive, being in accordance with Mehra and Prescott (1985) hypothesis.

The literature does not predict homogeneous results for the CRRA coefficient as there are several factors that might influence the coefficient estimation: Expiration time of the contracts, length of estimation period, among other factors cause the authors to present conflicting results. Bliss and Panigirtzoglou (2004) report an average value for the CRRA coefficient estimated from 3-week contracts to be 6,85. Conversely, Normandin and St-Amour (1998) estimate risk aversions based on the NYSE index to gauge the post-war equity premium concluding that risk aversion generally assumes values below 3. While Smith and Whitelaw (2009) estimate a CRRA coefficient of 3,33. The obtained results are in line with the literature expectations. Table 9 reports the estimated coefficients for the crisis period.

Table 9

This Table reports the coefficients for the constant relative risk aversion (CRRA) along with the sum of the log-likelihood function. The coefficients were estimated from the ten Mixture of Lognormal distributions (MLN) risk neutral densities estimated beforehand. The time frame considered ranged from November 22nd, 2008 (First expiration date) to September 19th, 2009 (Last expiration date).

Crisis	Expected Utility	Rank-Dependent
ϒ	0,47	-1,94
Σ Log-Likelyhood	-54,82	-54,48

In the post crisis period, EU CRRA coefficient reports low risk aversion while RDEU reports negative risk aversion (i.e. investors behave as risk-seeker agents). Smith and Whitelaw (2009) propose that generally increases during periods of financial crisis. Conversely, comparing both periods' results, the coefficients seem to hint for a decrease in risk aversion.

Jakwerth (2000) finds negative risk aversion coefficients after the 1987 crash. Moreover, the author considers that fundamental mispricing in the derivatives market to be the main driver for such estimate. It might be reasonable to pose that in the months prior to the crisis, a consistent mispricing in the derivatives market could lead to the CRRA coefficient to assume values bellow the historical average values (i.e. considering Bliss and Panigirtzoglou (2004) 3-week contracts reference value); namely, even negative values, justifying the findings reported in this study.

Additionally, Bliss and Panigirtzoglou (2004) indicate that the risk aversion coefficient is sensitive to volatility. The authors explain that in periods of high volatility, the risk averse investors tend to leave the market. As a result, the agents that remain in the market (i.e. those impacting the distributions derived from option prices) are more prone to risk.

Following the authors reasoning, and considering the evidence above for the increased volatility during the crisis period, it would be reasonable to consider that the two estimated parameters for the crisis period that score significantly below the historical average could be explained by the large selling pressure felt during the subprime crises driving out risk averse investors, decreasing the general level of risk aversion in the market, even if momentarily.

Finally, Liu et al. (2007) reports that by including periods of financial turmoil in a sample for the FTSE 100 during the dot.com bubble, the estimated CRRA coefficients obtained were significantly lower than Bliss and Panigirtzoglou (2004). The fact that periods of recession were included in the pre-crisis sample might have lowered the coefficients. From table 8 and 9, the RDEU was able to achieve higher sum of log-likelihood when compared to the EU; thus, interpretation should be drawn from RDEU as it has more explanative power.

Regarding the statistical significance of the coefficients, the Likelihood-ratio test was conducted using the wilk's theorem that approaches the distribution of the likelihood ratio to a chi-square distribution, being the degrees of freedom, the number of restrictions imposed to the model. For the EU CRRA a chi-square with one degree of freedom was used (i.e. the restriction used was ϒ=0) while for the RDEU CRRA, three degrees of freedom were used (i.e. the restrictions were ϒ=0, α=1 and β=1). Wilk's theorem approaches the likelihood ratio to a chi-square when the number of months used to estimate tends to infinity. Table 10 summarizes the results of the Likelihood-ratio test.

Table 10

This table reports the sum of the log-likelihood function obtained in the estimation of the risk aversion coefficients for both the EU and RDEU and for both the pre-crisis and crisis period. The test statistic should be tested against a χ^2distribution. The degrees of freedom considered for the EU is 1 and for the RDEU 3. For the EU, the critical values are 3,85 and 2,81 for 5% and 10% significance levels respectively, while for the RDEU the critical values are 7,81 and 6,25 for 5% and 10% significance levels respectively.

Pre-Crisis	Expected Utility	Rank-Dependent
Σ Log-Likelyhood	-57,37	-55,48
Test Statistic	0,73	4,51

Crisis	Expected Utility	Rank-Dependent
Σ Log-Likelyhood	-54,82	-54,48
Test Statistic	0,02	0,70

Under this test, the estimated coefficients fail to be significant for the significant values considered. The rank dependent coefficient for the pre-crisis period is significant for a 25% significance level (i.e. tested against a critical value of 4,11). The drivers for failing the likelihood-ratio test might be either the procedure suggested by Bliss and Panigirtzoglou (2004) not being appropriate to this data set. In fact, by using such a low number of months (i.e. 10 months), the likelihood-ratio might fail to follow a chi-square distribution as Wilk's theorem might not be applicable for such a low number of estimation periods. On the other hand, the mixture of lognormal distributions methodology may not be able to capture the real-world densities properly for this data set.

Real world Densities

To estimate the real-world densities, the transformed probabilities (p*) and prices (S1* and S2*) were applied to the previous estimated RND. Table 11 presents the descriptive statistics as well as the performance of the model.

Table 11

This Table reports the average values for first four central moments of the distribution: Mean, Standard Deviation, Skewness and Kurtosis. Moreover, this table summarizes the quality assessment regarding the Real-World densities estimated with the mixture of lognormal distributions under the RDEU. Total $G(\theta)$ represents the average of the aggregated mispricing computed for each month, Average represents the average mispricing per option, Minimum and Maximum represent the average minimum and maximum $G(\theta)$ obtained for each month and standard deviation represents the average standard deviation of the $G(\theta)$. The time frame considered was from September 22, 2007 until August 22, 2009.

Moments	Pre-crisis	Crisis
Mean	1 444,30	858,82
Standard Deviation	51,93	75,49
Skewness	-0,39	-0,70
Kurtosis	3,21	4,91

Average G(θ)	Pre-crisis	Crisis
Total	7 409,04	12 714,96
Average	81,22	139,42
Minimum	2,67	6,46
Maximum	146,86	259,74
Standard Deviation	56,62	87,26

Comparing table 11 with tables 1 to 4 (MLN sections only), allows for a better understanding of how the real-world transformation impacted the densities. Regarding the central moments of the distribution, and for the Pre-crisis period, on average, the mean value increased while the standard deviation decreased. Moreover, the skewness became less negative and the kurtosis increased. Figure 5 depicts the last density of the pre-crisis period but reflects the described average changes occurring during the period. The shift to the right is visible, the bi-modal distribution is less accentuated, the right tail seems shorter and the distribution is more peaked.

Concerning the crisis period, the mean value decreased while the standard deviation increased. The skewness became less negative but the kurtosis increased. Unlike in figure 5, figure 6, depicting the first estimation month's density, does not represent the average changes so well. The shift to the left is visible (i.e. reduction of the mean) but the skewness becomes slightly more negative while the kurtosis decreases. Appendix 2 presents the remaining distributions for comparability between the RND.

The findings are in line with the literature as the distribution tends to be less skewed with the transformation for the real-world densities Liu et al.(2007), have lower standard deviation and the fact that there is opposite evolutions in the two periods seems to be consistent with the fact that the risk aversion parameter suffered a signal (and relevant absolute) change, Moreover, the rank dependent utility function had a better performance during the pre-crisis period considering all measured of mispricing.

Figure 5

This figure depicts the impact of the real-world transformation. The black line represents the RWD curve while the dotted line represents the RND. The density is respective to the expiration date of June 21, 2008.

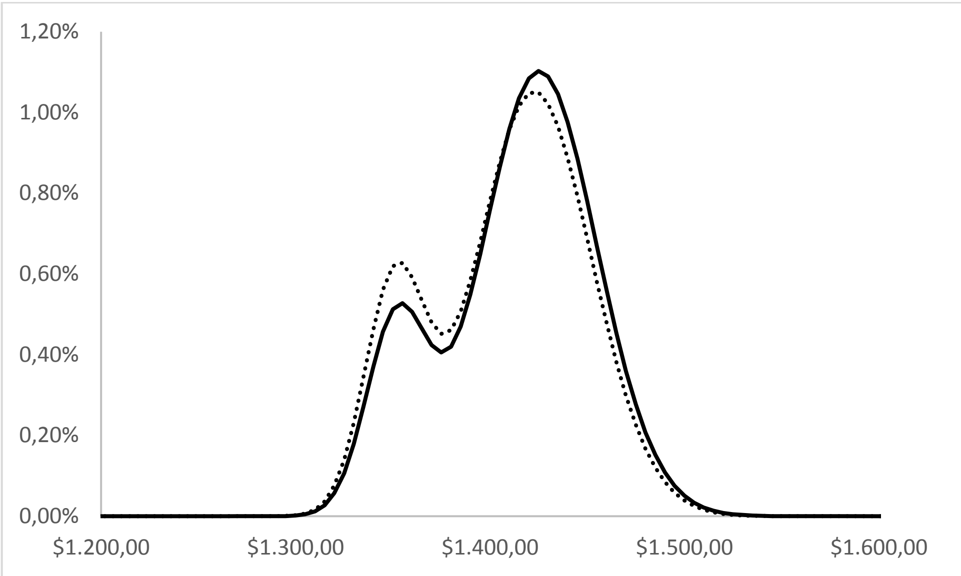
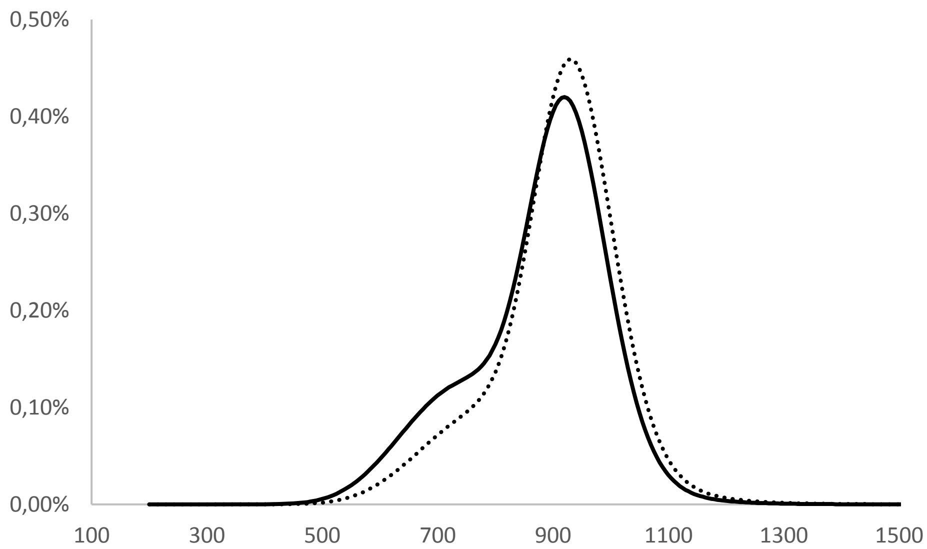


Figure 6

This figure depicts the impact of the real-world transformation. The black line represents the RWD curve while the dotted line represents the RND. The density is respective to the expiration date of November 22, 2008.



Conclusions

RNDs estimated from option prices are considered reliable forecast tools, providing relevant information on the underlying asset distribution. From the three methods employed in the estimation of option implied RND, all performed reasonably well and produced results consistent with the literature for both samples considered. Overall, the mixture of lognormal distributions performed better than the GB2 and the Lognormal-Polynomials, by achieving the lowest mispricing error in both periods. The Lognormal-Polynomials' first estimation yielded negative densities on both samples; however, by imposing restrictions of the skewness and kurtosis level, the distributions were corrected, complying with the non-negative condition.

RND estimated during the crisis period had on average lower mean and higher standard deviation. Moreover, on average the distributions were more negatively skewed and the kurtosis was higher. The findings are in line with what the literature describes and provide a sensible picture of the state of the financial markets.

The sample considered here as pre-crisis included some recession periods and in a *post hoc* analysis there were already some indicators of financial instability even before the considered sample. Nevertheless, the markets rebounded from the Bear Stearns bankruptcy and before the failure of Freddie Mac and Fannie Mae, the market did not seem to reflect the crisis that was to come.

Relatively to the crisis sample, estimations reflect the effects of the financial crisis by having a much lower mean, in line with the trading levels verified at the time and have the particular characteristic of providing more similar shapes between the three models with lower pricing errors.

In a general way, the lognormal-polynomials produced distributions with positive skewness, oppositely to the mixture of the lognormal distributions and GB2. Moreover, all methodologies reported kurtosis levels equal or in excess of 3 while the standard deviation for the pre-crisis sample ranged around \$55 and \$67 for the crisis sample.

The estimation of the CRRA coefficient was conducted under the mixture of lognormal distributions methodology for both expected utility and rank-dependent expected utility. The coefficients were estimated using the complete sample of 10 months per period considered, assuming the representative investor's utility function to be power utility. For the pre-crisis sample, CRRA coefficients of 2,81 (EU) and 4,41 (RDEU) were obtained, being in line with the parameters described in the literature. For the crisis sample, CRRA coefficients of 0,47 and -1,94 were obtained. The literature both confirms and contradicts the findings. Some authors present evidence that in periods of financial crisis, the risk aversion decreases as the risk averse investors leave the market. Other authors report that the risk aversion is negatively correlated with the economic cycle, being then expected a higher CRRA coefficient in the crisis sample.

In the literature, there is no consensus regarding the CRRA coefficient, being reported that this estimation depends highly on the sample used, on the maturity of the contracts and on the method used for estimation. Nevertheless, in the sample considered in this dissertation, there is evidence for a decrease in the risk aversion coefficient; albeit, the fail to pass the likelihood-ratio test. The low number of estimation months considered might cause the failure of the test. Furthermore, other models could be employed to estimate the risk aversion coefficient for a more robust analysis.

For further research, it would be interesting to expand the sample, obtaining more estimation months and to consider put options to have more strike prices for each month of estimation. Moreover, future research could explore the implication of considering exponential utility function instead of the power utility function as well as estimate the CRRA coefficients with different methodologies to make comparison possible.

Appendix

Appendix 1

This figure is extracted from Bookstaber and McDonald (1987) and represents the special cases of the Generalized Beta Distribution (GB2).

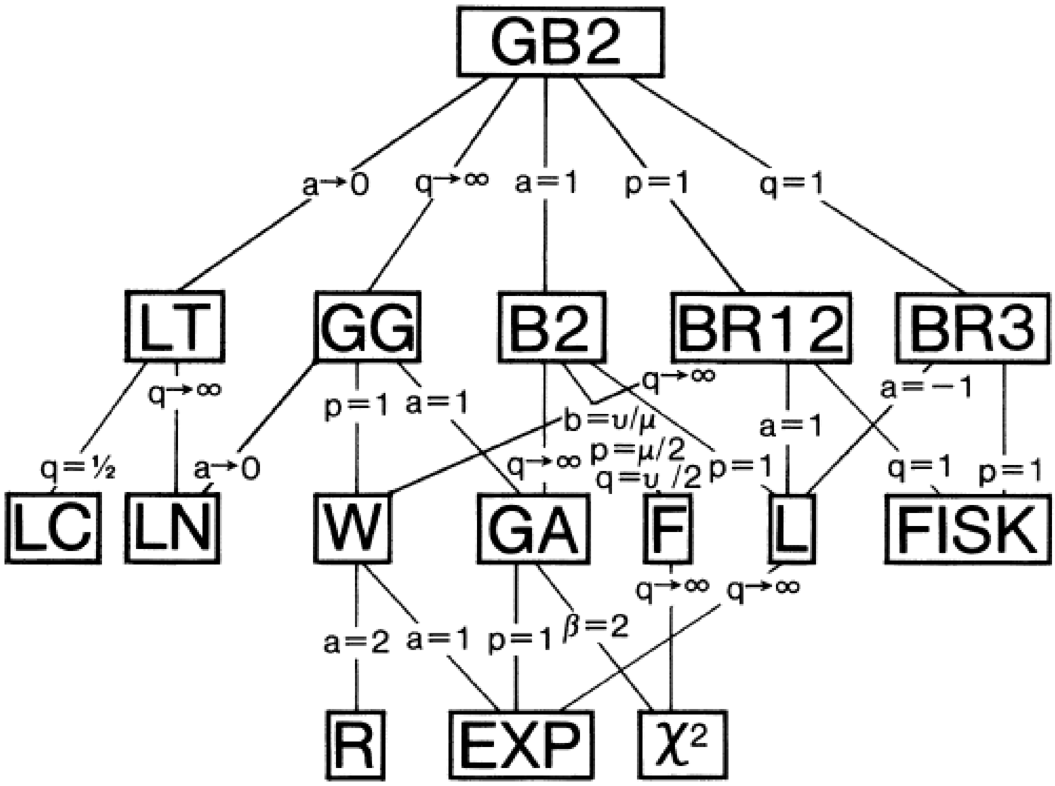
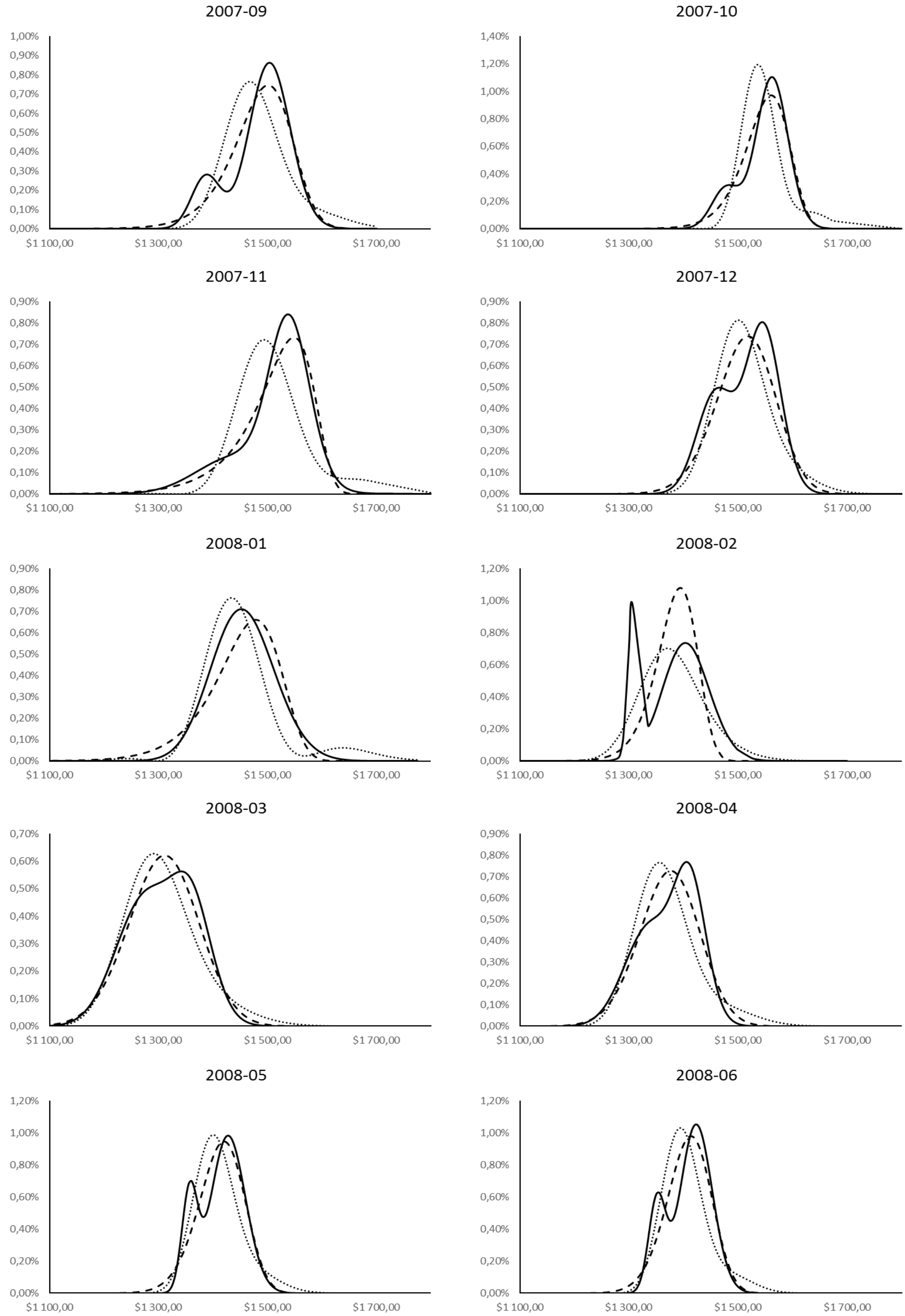


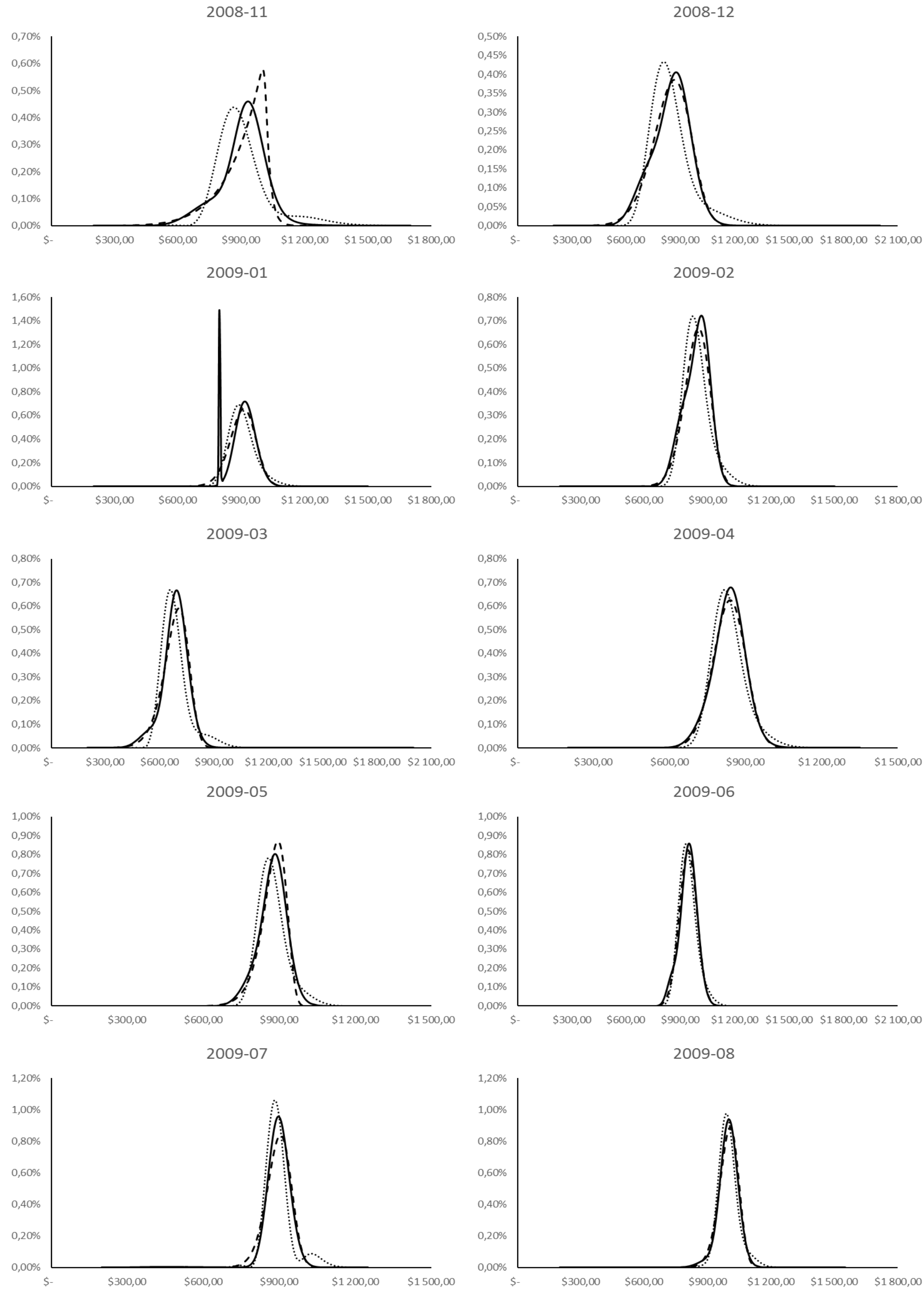
FIG. 1.—Distribution family tree. LT = log *t*; B2 = beta of the second kind; BR12 = Singh-Maddala or Burr type 12; BR3 = Burr type 3; LC = log Cauchy; LN = lognormal; W = Weibull; GA = gamma; L = Lomax; R = Rayleigh; EXP = exponential.

Appendix 2

This figure depicts the monthly RNDs estimated for the pre-crisis sample. The black line represents the RND extracted from the mixture of lognormal distributions. The dashed line represents the RND extracted from the GB2. The dotted line represents the RND extracted from the Lognormal-Polynomial.



This figure depicts the monthly RNDs estimated for the crisis sample. The black line represents the RND extracted from the mixture of lognormal distributions. The dashed line represents the RND extracted from the GB2. The dotted line represents the RND extracted from the Lognormal-Polynomial.



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