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# Sovereign debt, fiscal policy, and macroeconomic instability

**Francesco Carli**[1] | **Leonor Modesto**[2]

[1]Department of Economics, Deakin University, Geelong, Victoria, Australia

[2]Católica Lisbon School of Business and Economics, Universidade Católica Portuguesa, Palma de Cima, Lisbon, Portugal

**Correspondence**
Francesco Carli, Department of Economics, Deakin Businees School, 70 Elgar Rd, Burwood, Victoria, Australia.
Email: francesco.carli@gmail.com

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**Abstract**

We study the relation between capital accumulation, fiscal policy, and sovereign debt dynamics in a small open economy. The government maximizes spending, facing borrowing constraints and a conditionality requirement. Debt dynamics are forward looking, being driven by the endogenous borrowing constraint. Current debt is determined by expectations about the government's ability to finance itself in the future, opening the room for indeterminacy. If the government believes it may issue more debt next period, the borrowing constraint relaxes, and current debt increases. The government invests more in productive activities, generating a boom which increases tax revenues. However, as this increase does not repay the additional debt, the government will issue more debt next period, confirming initial expectations. To exclude explosive trajectories, tax revenues must increase enough to repay the outstanding debt and reduce future debt emission. This is possible only with a sufficiently procyclical tax rate. In this case, if productive externalities are large enough, the economy exhibits local and global indeterminacy, as steady-state multiplicity is also obtained. Avoiding sufficiently procyclical tax rates, the government can prevent local and global fluctuations driven by self-fulfilling volatile expectations. This differs from the general

wisdom that procyclical tax rates should be used for stabilization.

## 1 | INTRODUCTION

The consequences of sovereign debt for government solvency and economic stability received a lot of attention in recent years. This interest has been reinforced by several episodes where countries experienced debt management problems and required the intervention of international financial institutions.[1] These episodes refueled the debate on debt sustainability, and the interest in understanding the dynamics of debt and their consequences on macroeconomic stability.[2] In this paper we contribute to this literature and show that governments' limited commitment to repay can generate indeterminacy and endogenous fluctuations driven by self-fulfilling volatile expectations. The (local) indeterminacy mechanism considered is novel and, in contrast to most papers,[3] is not related to the working of the labor market. Further, we show that conventional stabilization policy recommendations may no longer apply.

We investigate these issues in a dynamic macroeconomic model for a small open economy built on four blocks. First, we consider a small open economy with an underdeveloped financial sector, where sovereign debt is held by foreign lenders.[4] Our setup applies to countries that, having experienced difficulties in repaying external loans, are subject to programs of intervention by international institutions.

Second, we assume that sovereign debt is subject to limited enforcement and that countries can renege on their obligations. Contractual obligations cannot be enforced with a sovereign the same way they are enforced with a private individual, and debt issuance is subject to a limited commitment problem.[5] There are however costs associated with default: creditors may retaliate, rendering access to new credit difficult. In addition, upon default a government will suffer a loss of reputation, which will also limit future access to international credit markets.[6] International lenders are aware that the government may default. Hence, they will impose a no-default constraint that generates endogenous borrowing limits, so that in equilibrium there will be no default.[7]

Third, to address the commitment problem and alleviate incentive issues, international loans are subject to a conditionality requirement. Countries in crisis are often in poor economic conditions because of bad policy choices in the past. Hence, policy reforms are required to speed up economic recovery.[8] We model these requirements by assuming that creditors request

[1]For example, since the end of 2009 Greece, Portugal, Spain, and Ireland were unable to repay their government debt and required the assistance of third parties like the European Central Bank (ECB) or the International Monetary Fund (IMF).

[2]See Collard et al. (2015) and Gourinchas et al. (2020) among others.

[3]See, for example, Benhabib and Farmer (1994), Schmitt-Grohé and Uribe (1997), Chen and Guo (2013), Sorger (2018), and Huang et al. (2019).

[4]Muller et al. (2019), in a framework intended to discuss the dynamics and renegotiations of sovereign debt during economic downturns, and Cole and Kehoe (2000) also assume that all debt is held by foreigners.

[5]Some references are Eaton and Gersovitz (1981), Bulow and Rogoff (1989), Kehoe and Perri (2002), Aguiar and Gopinath (2006), and Arellano (2008).

[6]See Thomz and Wright (2010).

[7]For a theoretical work with exogenous borrowing constraints see Le Van and Pham (2017).

[8]Conditionality requirements have been extensively adopted by the IMF and the World Bank as part of their lending programs. According to the IMF “When a country borrows from the IMF, its government agrees to adjust its economic

governments to invest a positive fraction of the resources received in production-enhancing activities.[9] We assume that this investment directly enters the production function as a productive externality.

Fourth we assume that the government wants to maximize spending, similarly to Acharia and Rajan (2013) and Bocola and Dovis (2019). Specifically, the government's objective is to maximize its probability of reelection, and it does so maximizing public spending which enters households' utility. To finance its expenditures the government can raise money either by taxing domestic residents or by borrowing in the international credit market by issuing one-period bonds. As the government wants to maximize spending, it will borrow as much as possible, regardless of its fiscal capacity.[10] This is consistent with the literature that relates sovereign borrowing behavior to political pressure for wasteful spending (Talvi & Vegh, 2005).

More formally, we consider a small open economy with two types of households: hand-to-mouth households and capitalists. Both derive utility from private consumption and government services. As in Arellano and Bai (2017) and Cole and Kehoe (2000), households do not directly participate in financial markets. Hand-to-mouth households do not save and provide labor inelastically,[11] whereas capitalists do not work and save in the form of capital.[12] Firms produce the domestic final good using capital and labor. We assume neither labor nor productive capital international mobility. Since the government maximizes public spending that enters households' utility, both the no-default constraint and the conditionality requirement bind at equilibrium.

The dynamics of the model are determined by the accumulation of productive capital and the evolution of sovereign debt. Debt dynamics are driven by the endogenous borrowing constraint, and in particular debt issued in period $t$ is determined by expectations about the government's ability to issue debt and to collect taxes in the future.[13] Thus, debt is a forward-looking variable, which opens the room for local equilibrium indeterminacy and therefore expectation-driven fluctuations.

Local indeterminacy emerges because if the government becomes confident that it will be easier to issue debt in the future, the borrowing constraint relaxes today and current debt also increases. To satisfy the conditionality requirement, the government invests more resources in production-enhancing activities. As a result, the marginal productivity of capital and labor increases, as well as output and after-tax income. As capitalists' savings increase, so does next period capital. This in turn increases future income and future tax revenues. However, the additional tax revenues are not enough to repay the additional debt issued. The government needs to issue additional debt the following period, which confirms initial expectations. However, to exclude explosive debt trajectories, the economy will have to return to the steady state in the absence of further shocks on expectations. This means that tax revenues need eventually to become large enough for the government to repay

policies to overcome the problems that led it to seek financial aid from the international community. These loan conditions also serve to ensure that the country will be able to repay the Fund so that the resources can be made available to other members in need."

[9]Muller et al. (2019) also consider the need for structural reforms, introducing them as part of an optimal contract between a sovereign borrower and lenders.

[10]This means that there is no trade-off between taxation and borrowing.

[11]As the indeterminacy mechanism is not related to the labor market to simplify the model we do not introduce a labor supply decision.

[12]Similarly to us Cole and Kehoe (2000) assume that consumers cannot borrow from nor lend to international bankers, but can accumulate capital which is held only by them. Arellano and Bai (2017), in a model with equilibrium default, assume that consumers do not save nor borrow. Thus, all domestic agents are hand-to-mouth households in their model.

[13]We obtain autonomous debt dynamics, in contrast to Nishimura et al. (2015) and Onori (2018) who impose a maximum debt to gross domestic product (GDP) ratio. Note that this is equivalent to imposing a mechanical debt limit which renders debt dynamics identical to the dynamics of GDP.

outstanding debt and reduce future debt emission. This is only possible with a sufficiently procyclical tax rate.[14] So, in contrast with most of the previous results on the effects of taxation on local stability, we find that local indeterminacy requires a sufficiently positive elasticity of the tax rate function with respect to income.[15] Indeed, the indeterminacy mechanism, which relies on the existence of borrowing constraints due to limited commitment to repay, is totally different from the traditional mechanism where indeterminacy requires countercyclical/regressive taxation, which operates through the choice between consumption and leisure.[16]

To understand the dynamic properties of the model we study global dynamics as steady-state multiplicity easily emerges in our model. This feature is related to the properties of the tax function: the greater the cyclical response of the tax rate, the greater is the likelihood of multiple steady states. If the tax rate response to the business cycle is not too positive, the steady state is either unique (a saddle) or there exist two steady states (a saddle and a source). However, in both cases we have neither global nor local indeterminacy. Differently, if the tax rate is sufficiently procyclical and the productive externality is not too small, we always have steady-state multiplicity. The high output steady state is a saddle but the low output steady state is now a sink, hence a global attractor. In this case we have both local and global indeterminacy.[17] Therefore there exist local stochastic endogenous fluctuations around the sink steady state (sunspots).[18] Moreover, since depending on expectations, the economy may either converge to the low or to the high output steady state, there exists a regime-switching rational expectations equilibrium, where the economy jumps between paths converging to different steady states according to a sunspot variable. We conclude that by choosing tax rate rules that are not too procyclical, the government can prevent local or global fluctuations driven by self-fulfilling volatile expectations. This result differs from the general wisdom that procyclical tax rates should be used for stabilization purposes.[19]

## 1.1 | Related literature

Our paper builds on the literature that studies the relation between sovereign debt dynamics and fiscal policy. Arellano and Bai (2017) propose a quantitative model to study the interactions between fiscal restrictions and government borrowing and default. We abstract from default, and focus instead on the emergence of endogenous fluctuations driven by volatile expectation about the government's future ability to issue debt. Moreover, we study the interplay between capital and debt dynamics, whereas in Arellano and Bai (2017) the accumulation of capital is not present.

Like us, Nishimura et al. (2015) show that constraints on public debt can be a source of local indeterminacy, and therefore, of expectations-driven fluctuations, in a model with capital accumulation. However they consider a closed economy where the government needs to satisfy a debt-to-GPD ratio, so that debt dynamics are pinned down by GDP dynamics. We introduce instead an

[14]Note that, in macromodels, procyclical and progressive tax rates are similar in terms of the stability properties of the equilibrium.

[15]We also find that a sufficiently high elasticity of the production externality is needed for the emergence of local indeterminacy.

[16]Note that in our framework, where households provide inelastically one unit of labor and do not smooth consumption this last mechanism is completely shut down.

[17]Global indeterminacy means that for the same initial value of the predetermined variable (capital) there are different equilibrium trajectories that converge to different steady states.

[18]See Grandmont et al. (1998).

[19]See, for example, Guo and Lansing (1998).

endogenous borrowing constraint so the joint dynamics of sovereign debt and capital determine the evolution of GDP and other economic variables. In both models, distortionary taxation is a key ingredient. However, we find that procyclical tax rates are required for local indeterminacy, while in their case tax rates are countercyclical. In our model the strength of productive externalities is linked to the ability of the government to raise income. This is the channel through which more progressive tax rates imply higher productive externalities, promoting local indeterminacy and therefore expectation-driven local fluctuations.[20] Moreover, we show that procyclical tax rates also generate global indeterminacy so that the economy, in response to expectation shocks, may switch between trajectories converging to different steady states, which may result in fluctuations considerably larger than fluctuations along trajectories converging to the same steady state.

Corsetti et al. (2014) also study the relation between fiscal policy and instability caused by self-fulfilling expectations in a two-country model with sovereign debt. Their main focus is on the sovereign risk channel, that is the transmission of sovereign risk to private sector borrowing conditions. We abstract from this channel, as the private sector does not participate in financial markets.

Cole and Kehoe (2000) consider a framework where, like in ours, the government's inability to commit to repay may generate multiple equilibria. However, in contrast to us, in their model self-fulfilling debt crises where default occurs are possible, and they focus on the optimal policy response to the threat of such crisis.

Finally note that, although we borrow some elements of the government behavior from political theory models, we do not provide in this paper a political economy theory of public spending, taxation, and debt, nor is this a paper on optimal taxation. Our main focus is on the dynamic implications of the inability of the government to fully commit to repay on the emergence of macroeconomic fluctuations driven by self-fulfilling expectations and on how to avoid the costs of these fluctuations.

The rest of the paper is organized as follows. Section 2 presents the model. The perfect foresight equilibrium is obtained in Section 3. Steady-state existence and multiplicity are discussed in Section 4. Section 5 analyzes local dynamics and discusses the local indeterminacy mechanism. Global dynamics are presented in Section 6. In Section 7 we discuss the role of fiscal policy on the local and global properties of our dynamic system. Finally we provide some concluding remarks in Section 8. Proofs and computations are relegated to the appendix.

## 2 | THE MODEL

Time is discrete and lasts forever: $t = 1, 2, \ldots$. We consider a small open economy with four types of agents: (i) firms, (ii) hand-to-mouth households, (iii) capitalists, and (iv) the government. The population size of hand-to-mouth households and that of capitalists are constant and each normalized to one.

[20]Chen and Guo (2013), in a closed economy with no debt, also find that sufficiently procyclical tax rates generate local indeterminacy. In their model, like in ours, the higher the ability of the government to raise income the higher will be productive externalities. Note however that in their paper the necessary and sufficient condition for indeterminacy is related to the slopes of the labor supply and labor demand schedules, as in Benhabib and Farmer (1994), and therefore works through the choice of consumption and leisure, which is absent in our model.

## 2.1 | Firms

Firms produce the final good using a Cobb–Douglas technology

$$y_t = Ai_t^{\varepsilon} k_t^{s} l_t^{(1-s)}, \tag{1}$$

where $k_t$ is the capital available for production at period $t$, $l_t$ is the labor, $s$ is the capital share, $A$ is a normalizing constant, $\varepsilon \geq 0$ is a parameter, and the function $i_t^{\varepsilon}$ represents production-enhancing activities generated from productive government expenditures $i_t$.

Firms maximize profits taking as given the real rental rate of capital $\rho_t$ and the real wage $w_t$, that is, they solve the following problem:

$$\max_{(k_t, l_t)} Ai_t^{\varepsilon} k_t^{s} l_t^{(1-s)} - \rho_t \cdot k_t - w_t \cdot l_t.$$

Optimality requires

$$Ai_t^{\varepsilon} s \left(\frac{k_t}{l_t}\right)^{s-1} = \rho_t, \tag{2}$$

$$Ai_t^{\varepsilon} (1-s) \left(\frac{k_t}{l_t}\right)^{s} = w_t. \tag{3}$$

## 2.2 | Hand-to-mouth households

Hand-to-mouth households supply in each period one unit of labor inelastically, earning the competitive wage $w_t$. They are taxed at the rate $\tau(y_t)$, which they take as given. Their preferences are defined over a composite good $g_t^{\zeta} \cdot c_t^{W}$, where $g_t$ is government spending, $c_t^{W}$ is their consumption, and $\zeta > 0$ measures the degree of public spending externalities on preferences. They do not save, consuming all their disposable income so that $c_t^{W} = (1 - \tau(y_t))w_t$.[21]

## 2.3 | Capitalists

Capitalists have discount factor $\beta$, and a log utility function. They do not work, and save in capital $k_{t+1}^{C}$. As hand-to-mouth households, capitalists get taxed at the rate $\tau(y_t)$ on the income earned on capital, which they take as given. We introduce again government spending positive externalities on preferences, so that the problem they solve is

$$\begin{aligned} &\max_{\left(c_t^{C}, k_{t+1}^{C}\right)} \sum_{t=0}^{\infty} \beta^t \log\left(c_t^{C} \cdot g_t^{\zeta}\right) \\ &\text{s.t. } c_t^{C} + k_{t+1}^{C} \leq [1 - \delta + (1 - \tau(y_t))\rho_t] k_t^{C}, \end{aligned} \tag{4}$$

[21]The property that hand-to-mouth households do not save in equilibrium can be microfounded, at a steady state, as in Becker (1980) and in Woodford (1986), by assuming that these agents discount the future more than capitalists.

where $c_t^C$ denotes the consumption of capitalists at period $t$. Solving the capitalists' problem we obtain[22]

$$c_t^C = (1-\beta)[1-\delta+(1-\tau(y_t))\rho_t]k_t^C, \tag{5}$$

$$k_{t+1}^C = \beta[1-\delta+(1-\tau(y_t))\rho_t]k_t^C. \tag{6}$$

## 2.4 | Government

We abandon the assumption of a benevolent government and, following Acharia and Rajan (2013) and Bocola and Dovis (2019), we consider instead an impatient government that maximizes expenditures which increase utility, $g_t$, pleasing both hand-to-mouth households and capitalists. Besides providing $g_t$, the government purchases production-enhancing goods $i_t$. Both these expenditures are financed by (i) issuing one-period bonds in the international bond market, $b_{t+1}$, taking as given the gross international interest rate $1+r$, which is constant and (ii) by taxing both labor and capital income according to the tax rate policy function $\tau(y_t) \in (0,1)$, where $y_t$ is aggregate income at time $t$. We consider an isoelastic specification for the tax rule, that is, we assume that $\tau(y_t) = \tau y_t^{\phi}$, where $\tau \in (0,1)$ and $\phi \in R$ is the elasticity of the tax rate. The tax rate is procyclical when $\phi > 0$ and countercyclical when $\phi < 0$. When $\phi = 0$ the tax rate is constant and equal to $\tau$.

The government can default on international bonds: this is understood by international lenders that will impose borrowing constraints to prevent the government from defaulting. Moreover, international creditors require the government to invest at least a fraction $\alpha$ of the resources received in the international bond market into productive government spending $i_t$. Finally, we assume that the government is more impatient than households, because it may not be reelected. Specifically, the government discount factor is $\gamma < \beta$.

The government solves therefore the following problem:[23]

$$V_t(k_t, b_t) = \max_{(g_t, b_{t+1}, i_t)} g_t + \gamma V_{t+1}(k_{t+1}, b_{t+1}) \tag{7}$$

$$\text{s.t. } g_t + (1+r)b_t + i_t = \tau(y_t)[w_t + \rho_t k_t] + b_{t+1}, \tag{8}$$

$$i_t \geq \alpha b_{t+1}, \tag{9}$$

$$V_{t+1}(k_{t+1}, b_{t+1}) \geq \bar{V}_{t+1}(k_{t+1}, b_{t+1}), \tag{10}$$

where $r$ is the interest rate on international loans that the government takes as given, (8) is the government budget constraint, and (9) and (10) are, respectively, the conditionality requirement and the no-default constraint imposed by creditors.[24]

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[22]See Appendix A.1.

[23]This formulation of the government's objective function is similar to Arellano et al. (2017), where the government maximizes the present discounted value of utility that the agents derive from the public good, net of default costs. Since in our model default is an off-equilibrium event, equilibrium default costs equal zero.

[24]For simplicity, we assume that the government's utility function is linear. In Appendix A.2, we show that the dynamical system that characterizes the evolution of the economy when the government's utility function is concave is similar to the one that describes the evolution of the economy when the government's utility function is linear.

Constraint (9) captures the policy reforms that the government is required to undertake to speed up economic recovery, to access the loan from international lenders. Through this constraint, the government needs to adjust its economic policy to increase productivity, by investing a fraction of the loan received in productive-enhancing investment, ensuring therefore that the country will be able to repay in the future.

The no-default constraint (10) guarantees that the continuation value $V_{t+1}(k_{t+1}, b_{t+1})$ exceeds the deviation payoff $\overline{V}_{t+1}(k_{t+1}, b_{t+1})$. We model the deviation payoff in a parsimonious way, as in Gu et al. (2013), by considering a reduced form that captures the main drivers of the default decision:

$$\overline{V}_{t+1}(k_{t+1}, b_{t+1}) = (1-\chi)V_{t+1}(k_{t+1}, b_{t+1}) + \xi b_{t+1}.$$

On the one hand, by defaulting the government avoids future payments of both principal and interest of the debt issued. This gain is increasing in the amount of outstanding debt $b_{t+1}$; $\xi > 0$ parametrizes the temptation to default. On the other hand, upon default the government access to financial markets is restricted, that is, creditors may retaliate by denying access to new credit (Eaton & Gersovitz, 1981) or the loss of reputation may limit future loans (Thomz & Wright, 2010). We capture this cost by admitting that only a fraction $(1-\chi) \in [0, 1]$ of the continuation value is accessible.[25] In this way, constraint (10) can be rewritten as

$$\begin{aligned} & V_{t+1}(k_{t+1}, b_{t+1}) \geq (1-\chi)V_{t+1}(k_{t+1}, b_{t+1}) + \xi b_{t+1} \\ & \Rightarrow b_{t+1} \leq \omega \cdot V_{t+1}(k_{t+1}, b_{t+1}), \end{aligned} \tag{11}$$

where $\omega \equiv \frac{\chi}{\xi}$, so that Equation (11) defines an endogenous debt limit.[26] Specifically, we can interpret the right-hand side of (11) as the largest value of debt such that repayment is individually rational.[27] With this formulation the first-order conditions for optimality are

$$\lambda_t = 1 > 0, \tag{12}$$

$$-\lambda_t + \eta_t = 0 \quad ==> \quad \eta_t = \lambda_t > 0, \tag{13}$$

$$\gamma \frac{\partial V_{t+1}(k_{t+1}, b_{t+1})}{\partial b_{t+1}} + \lambda_t + \mu_t \left[\omega \cdot \frac{\partial V_{t+1}(k_{t+1}, b_{t+1})}{\partial b_{t+1}} - 1\right] - \alpha \cdot \eta_t = 0, \tag{14}$$

where $\lambda_t$, $\eta_t$, and $\mu_t$ denote, respectively, the Lagrange multipliers associated with (8), (9), and (10). From the envelope condition we obtain

$$\frac{\partial V_{t+1}(k_{t+1}, b_{t+1})}{\partial b_{t+1}} = -\lambda_{t+1}(1+r).$$

[25]We choose this specification for the deviation payoff to parsimoniously capture the margins relevant for the default decision, namely, the short-term benefit from reneging the debt obligation, $\xi b_{t+1}$, and the long-term cost from doing so, $(1-\chi)V_{t+1}(b_{t+1}, k_{t+1})$. Consistently with our specification of the deviation payoff $\overline{V}_{t+1}(k_{t+1}, b_{t+1})$, we assume $\xi > 0$.

[26]In Appendix A.3 we show that assuming, for example, that upon default the government is permanently excluded from international borrowing, would not change the essential features of the analysis. The reason is that, under this last assumption, the evolution of the economy would still be described by a system of two dynamic equations similar to (22) and (23).

[27]Note that constraint (11), through the value function $V_{t+1}(b_{t+1}, k_{t+1})$, implies a link between $b_{t+1}$ and $k_{t+1}$. The government takes as given the debt limit $\omega V_{t+1}(b_{t+1}, k_{t+1})$ when solving its maximization problem and, in equilibrium, such debt limit has to be consistent with no default.

Replacing this last expression together with (12) and (13) in (14) we get $\mu_t = \frac{(1-\alpha)-\gamma(1+r)}{1+\omega(1+r)}$.

## 3 | EQUILIBRIUM

One dynamic equation that describes the evolution of the system is the dynamic equation for capital that we obtain from the capitalist problem and the equilibrium in the capital market:

$$k_{t+1} = \beta[1 - \delta + (1 - \tau(y_t))\rho_t]k_t. \tag{15}$$

The second dynamic equation comes from the evolution of bonds and it is the solution of the following system:

$$\mu_t = \frac{(1 - \alpha) - \gamma(1 + r)}{1 + \omega(1 + r)}, \tag{16}$$

$$\mu_t[\omega V_{t+1}(k_{t+1}, b_{t+1}) - b_{t+1}] = 0, \tag{17}$$

$$g_t + (1 + r)b_t + i_t = \tau(y_t)y_t + b_{t+1}, \tag{18}$$

$$i_t = \alpha b_{t+1}. \tag{19}$$

In the rest of the analysis, we introduce the following assumption on the government discount factor.

**Assumption 1.** The government is sufficiently impatient: $0 < \gamma < \frac{1-\alpha}{1+r}$.

From Equation (16) and Assumption 1, it follows that $\mu_t > 0$. Hence, from (17), the government will borrow up to the point where the no-default constraint binds. Therefore, using (7), we can rewrite (11) as

$$b_{t+1} = \omega \cdot V_{t+1}(k_{t+1}, b_{t+1}) = \omega[g_{t+1} + \gamma V(k_{t+2}, b_{t+2})].$$

Now, as from (11) forwarded one period we have that $b_{t+2} = \omega \cdot V_{t+2}(k_{t+2}, b_{t+2})$, we finally obtain

$$b_{t+1} = \omega \cdot g_{t+1} + \gamma b_{t+2}. \tag{20}$$

Equation (20) simple states that, when the borrowing constraint is binding, debt issued in period $t$, $b_{t+1}$, equals the existing upper limit for debt emission today, $\omega \cdot V_{t+1}(k_{t+1}, b_{t+1}) = \omega \cdot g_{t+1} + \gamma b_{t+2}$. Substituting now the government budget constraint (18) and expression (19) in (20) we obtain

$$b_{t+1} = \frac{\omega}{1 + \omega(1 + r)}\tau(y_{t+1})y_{t+1} + \frac{\omega(1 - \alpha) + \gamma}{1 + \omega(1 + r)}b_{t+2}, \tag{21}$$

which is the second dynamic equation. This equation tells us that debt issued in period $t$, $b_{t+1}$, equals the existing upper limit for debt emission in that period, and that this upper limit is determined by expectations on the ability of the government to issue new debt in the following period, $b_{t+2}$, and on its capability of raising tax revenues next period, $\tau(y_{t+1})y_{t+1}$, that is, the existing upper limit for debt emission today is determined by expectations on the ability of the government to continue financing itself in the future. We conclude that, when the borrowing constraint binds, debt becomes a forward-looking variable.

Substituting (1) and (2) with $l_t = 1$, respectively, in (15) and (21), we can state the following:

**Definition 2.** Let Assumption 1 hold. An intertemporal equilibrium with perfect foresight is a sequence $(k_t, b_{t+1}) \in R^2_{++*}$, $t = 0, 1, 2, ..., \infty$, that, for a given $k_0 > 0$, satisfies

$$k_{t+1} = \beta\left[1 - \delta + \left[1 - \tau\left(A(\alpha b_{t+1})^{\varepsilon} k_t^{s}\right)^{\phi}\right] A(\alpha b_{t+1})^{\varepsilon} s k_t^{s-1}\right] k_t, \tag{22}$$

$$\omega\tau\left[A(\alpha b_{t+2})^{\varepsilon}\, k_{t+1}^{s}\right]^{1+\phi} + [\gamma + \omega(1-\alpha)]b_{t+2} - [1 + \omega(1+r)]b_{t+1} = 0. \tag{23}$$

Equations (22) and (23) represent, respectively, capital accumulation and the evolution of sovereign debt. They determine the dynamics of this economy through a two-dimensional dynamic system in $(k_t, b_{t+1})$, with only one predetermined variable, the capital stock $k$. Indeed, the amount of capital used in production at period $t$, $k_t$, is a variable determined by past savings of capitalists. Debt issued in period $t$, $b_{t+1}$, on the contrary, is affected by expectations of future debt limits and future ability to tax, and independent of the existing amount of debt at the beginning of the period. Therefore $b_{t+1}$ is not given by past variables, like $b_t$ and $k_t$, but determined by future variables, $b_{t+2}$ and $k_{t+1}$.[28]

Thus, when the government is sufficiently impatient, so that the endogenous borrowing constraint binds, debt is a forward-looking variable. Differently, as shown in Appendix A.4, when the endogenous borrowing constraint is slack, debt is a pre-determined variable and depends on past decisions.

## 4 | STEADY STATE

A steady-state solution $(k, b) \in \mathfrak{R}^2_{++}$ of the dynamic systems (22) and (23) is a stationary solution $k_{t+1} = k_t = k > 0$ and $b_{t+2} = b_{t+1} = b > 0$ of that system. Define $\theta \equiv 1 - \beta(1-\delta) \in (0, 1)$, then, using (1), steady-state values $(k, b) \in \mathfrak{R}^2_{++}$ of the dynamic systems (22) and (23), are solutions of the following equations:

$$(1 - \tau y^{\phi})A(\alpha b)^{\varepsilon}\, s k^{(s-1)} = \frac{\theta}{\beta}, \tag{24}$$

$$\frac{b}{\tau y^{(1+\phi)}} = \frac{\omega}{(1 - \gamma + \alpha\omega + r\omega)}, \tag{25}$$

$$\text{where } y = A(\alpha b)^{\varepsilon} k^{s}. \tag{26}$$

[28]Note however that, although $b_t$ does not influence the choice of $b_{t+1}$, it still determines the amount of current government expenditures, $g_t$. Indeed, using the government budget constraint (18) and expression (19) we have that $g_t = \tau(y_t)y_t + (1-\alpha)b_{t+1} - (1+r)b_t$.

Combining (24) and (26) we obtain

$$k = \frac{s\beta(1 - \tau y^{\phi})y}{\theta} > 0. \tag{27}$$

Solving now (25) for $b$ and substituting it and (27) in (26) we finally obtain the following definition:

**Definition 3.** Let Assumption 1 hold. Steady-state values $(k, b) \in \Re^2_{++}$ of the dynamic systems (22) and (23) and the corresponding level of output $y = A(\alpha b)^{\varepsilon} k^{s}$ are solutions of the following equations:

$$H(y) \equiv \frac{y}{\left[\frac{s\beta(1-\tau y^{\phi})y}{\theta}\right]^{s}\left[\alpha\frac{\omega\tau y^{(1+\phi)}}{(1-\gamma+\alpha\omega+r\omega)}\right]^{\varepsilon}} = A, \tag{28}$$

$$k = \frac{s\beta(1 - \tau y^{\phi})y}{\theta}, \tag{29}$$

$$b = \frac{\omega\tau y^{(1+\phi)}}{(1 - \gamma + \alpha\omega + r\omega)}. \tag{30}$$

For a given level of $y$, Equations (29) and (30) uniquely determine $k$ and $b$, respectively. However, a steady-state solution only exists if there is a value $y > 0$ satisfying Equation (28).[29] We ensure the existence of a steady state, namely, the normalized steady state $k_{nss}, b_{nss}$ with the corresponding level of output $y_{nss} \equiv 1$, by following the usual procedure of fixing the parameter $A$ at the appropriate level. It is easy to see that (28) is satisfied for the normalized steady state if and only if $A$ takes the value $A^* \equiv H(1)$. Hence, we deduce the following proposition.

**Proposition 4** (Normalized steady state). *Let Assumption 1 hold,* $k_{nss} = \frac{s\beta(1-\tau)}{\theta}$ *and* $b_{nss} \equiv \frac{\omega\tau}{(1-\gamma+\alpha\omega+r\omega)}$. *Then* $(k_{nss}, b_{nss})$ *with the corresponding level of output* $y_{nss} = A(\alpha b_{nss})^{\varepsilon} k_{nss}^{s} = 1$ *is the normalized steady state of the dynamic systems (22) and (23) if and only if* $A = A^* \equiv \frac{1}{(\alpha b_{nss})^{\varepsilon} k_{nss}^{s}} > 0$.

From now on we assume that $A = A^*$, so that the normalized steady state exists. However, nothing guarantees steady-state uniqueness. The number of steady states is determined by the number of intersections of the curve $H(y)$ with the horizontal line $A^*$. Of course, if the function $H(y)$ is always increasing or always decreasing, the steady state is unique. It follows that a necessary condition for steady-state multiplicity is that $H(y)$ is nonmonotonic. To further discuss this issue we will now obtain its elasticity:

[29] Note also that we have to guarantee that $\tau(y) = \tau y^{\phi} < 1$. This implies that we restrict $y \in (\underline{y}, \bar{y})$, where $\underline{y} = \left(\frac{1}{\tau}\right)^{\frac{1}{\phi}}$ and $\bar{y} = +\infty$ when the tax rate is countercyclical ($\phi < 0$), and with $\underline{y} = 0$ and $\bar{y} = \left(\frac{1}{\tau}\right)^{\frac{1}{\phi}}$ when the tax rate is procyclical ($\phi > 0$).

$$\epsilon_{H,y} \equiv \frac{H'(y)y}{H(y)} = [1 - \varepsilon(1 + \phi)] - s\left(1 - \phi\frac{\tau y^{\phi}}{1 - \tau y^{\phi}}\right). \tag{31}$$

Since $H(y) > 0$ and $0 < \tau y^{\phi} < 1$, we can see that $\epsilon_{H,y} > 0$ when $1 - \varepsilon(1 + \phi) - s > \tau y^{\phi}[1 - (\varepsilon + s)(1 + \phi)]$. Thus $\epsilon_{H,y}$ changes sign at most once. Since we have ensured the existence of the NSS this implies that we can have one or two steady states. It is also easy to see that when the tax rate is constant, that is, when $\phi = 0$, as $\epsilon_{H,y} > 0$ for every $y > 0$, there is only one steady state. In fact, the greater the response of the tax rate to income, the greater is the likelihood of steady-state multiplicity. As in Nishimura et al. (2015), Abad et al. (2020), and Lloyd-Braga and Modesto (2017) the emergence of multiplicity is related to the properties of the tax function.

## 5 | LOCAL DYNAMICS

In this section, we study the local stability properties of the dynamic systems (22) and (23) around an interior steady state $(k, b)$. We follow the usual procedure of linearizing the perfect foresight system around the steady state obtaining

$$\tilde{k}_{t+1} = \left(1 - \frac{\theta s \tau \phi}{1 - \tau} - \theta(1 - s)\right)\tilde{k}_t + \varepsilon\theta\left(1 - \frac{\tau\phi}{1 - \tau}\right)\tilde{b}_{t+1}, \tag{32}$$

$$\begin{aligned}&[\gamma + \omega(1 - \alpha) + \varepsilon(1 + \phi)(1 - \gamma + \alpha\omega + r\omega)]\tilde{b}_{t+2} + s(1 + \phi)(1 - \gamma + \alpha\omega + r\omega)\tilde{k}_{t+1} \\ &= [1 + \omega(1 + r)]\tilde{b}_{t+1},\end{aligned} \tag{33}$$

where a ~ over a variable denotes percentage deviations from the steady state, for example, $\tilde{k}_t = \frac{dk_t}{k}$.

We use a geometrical method (see Grandmont et al., 1998), which allows us to analyze how the trace, $T$, and determinant, $D$, of the Jacobian matrix of systems (32) and (33) (whose expressions are given in Appendices A.5 and A.6) or equivalently the local eigenvalues, evolve in the space $(T, D)$ when some relevant parameters of the model are made to vary continuously in their admissible range. In Figure 1 we have represented in the plane $(T, D)$ three lines relevant for our purpose: the *line AC* $(D = T - 1)$ where a local eigenvalue is equal to 1; the *line AB* $(D = -T - 1)$, where one eigenvalue is equal to $-1$; and the *segment BC* ($D = 1$ and $|T| < 2$) where two eigenvalues are complex conjugates of modulus 1. When $T$ and $D$ fall in in the interior of triangle $ABC$ the steady state is a *sink* (both eigenvalues with modulus lower than one). The steady state is a *saddle* (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than one) when $|T| > |D + 1|$, and a *source* (both eigenvalues with modulus higher than one) in the remaining regions. In the present context, where only one variable, capital, is predetermined, when the steady state is a sink, it is locally indeterminate.[30] This means that there are infinitely many bounded deterministic intertemporal equilibria, that eventually converge to the steady state. In this case there

[30]Indeterminacy occurs when the number of eigenvalues strictly lower than one in absolute value is larger than the number of predetermined variables.

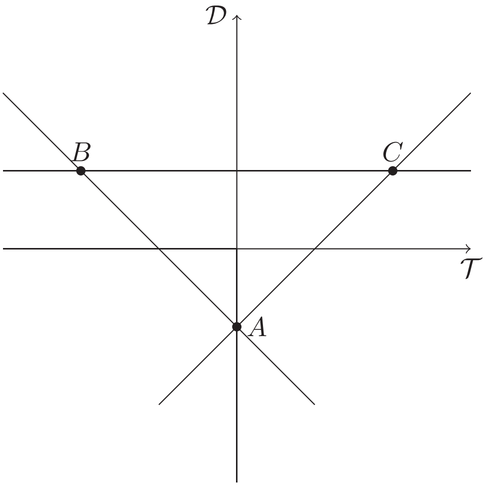


**FIGURE 1** $(T, D)$ space

are also infinitely many nondegenerate stochastic equilibria driven by self-fulfilling expectations (sunspots equilibria), that stay arbitrarily close to the steady state.[31]

We can also use the same diagram to study local bifurcations. When, by slightly changing a (bifurcation) parameter, the values of $T$ and $D$ cross the $AB$ line, a *flip* bifurcation occurs and deterministic cycles of period two appear. When the values of $T$ and $D$ cross the $AC$ line, a *transcritical* bifurcation occurs (one eigenvalue crossing the value 1). In this case, if $(T, D)$ is close enough to line $AC$, two close steady states, which exchange stability properties as $(T, D)$ crosses line $AC$, coexist.[32] Finally, when $T$ and $D$ cross the interior of the segment $BC$, a pair of complex conjugate eigenvalues crosses the unit circle and a Hopf bifurcation generically occurs. In this case there are deterministic cycles describing orbits that lie over an invariant closed curve, surrounding the steady state, in the state space. If the Hopf bifurcation is subcritical, this curve emerges when the steady state is a sink and sunspot fluctuations arbitrarily close to the steady state emerge.[33]

In our analysis we take $\beta$, $\omega$, $\alpha$, $\tau$, $\varepsilon$, $r$, $s$, $\gamma$, and $\phi$ as parameters characterizing our economy. We organize our discussion in terms of $\phi$ and $\varepsilon$ for fixed values of the other parameters. We first fix $\varepsilon$ and consider $\phi$ as the bifurcation parameter, by varying its value continuously in its domain. Furthermore we restrict our analysis to empirically plausible values of the parameters. We assume that steady-state after-tax income $(1 - \tau(y))y$ is increasing in income, that is, $\phi < \phi_c \equiv \frac{1-\tau}{\tau}$, and that steady-state tax revenues are nondecreasing in income, that is, $\phi > -1$. We also restrict our analysis to the case of not too strong distortions, considering that externalities are not too high. That is, $\varepsilon < 1$, and that tax rates are not too big $\tau < 1/2$. Also, since calibrated values for the capital share of output are usually lower than 1/2, and since values for $\theta \equiv 1 - \beta(1 - \delta)$ are rather small when the period is short, we assume that $1/2 > s > \theta(1 - s)$, as typically done in the literature, and also that

[31]See Grandmont et al. (1998).

[32]When $(T, D)$ is on line $AC$ the two steady states collapse into only one.

[33]When the Hopf bifurcation is supercritical the invariant closed curve appears when the steady state is a source and, although sunspot equilibria that stay arbitrarily close to the steady state do not exist, there are nevertheless infinitely many equilibria exhibiting bounded stochastic fluctuations around the invariant closed curve. See Grandmont et al. (1998).

$\theta(1-\tau-s)[1+\omega(1+r)] < s(1-\tau)(1-\gamma+\alpha\omega+r\omega)$. All these assumptions are summarized in Assumption 5, and we assume that they are satisfied in the rest of the paper.

**Assumption 5.** Small distortions, short period, and capital share of output small

(i) $0 \le \varepsilon < 1$,
(ii) $0 < \tau < 1/2$,
(iii) $-1 < \phi < \phi_c \equiv \frac{1-\tau}{\tau}$,
(iv) $0 < s < 1/2$,
(v) $\theta < \min\left[\frac{s}{1-s}, \frac{s(1-\tau)(1-\gamma+\alpha\omega+r\omega)}{(1-\tau-s)[1+\omega(1+r)]}\right]$.

*We summarize our local stability results in Proposition 6. The proof is provided in Appendix A.7.*

**Proposition 6.** *Let Assumptions 1 and 5 be verified and define*

$$\varepsilon_c = \frac{s\tau(1-\gamma+\alpha\omega+r\omega)}{s(1-\gamma+\alpha\omega+r\omega)-\theta[1+\omega(1+r)][(1-\tau)(1-s)+s\tau]}, \quad \phi_c = \frac{1-\tau}{\tau}, \quad \phi_g = \frac{(1-\tau)[\varepsilon-(1-s)]}{s\tau-\varepsilon(1-\tau)},$$

*and*

$$\phi_h = \frac{(1-\tau)\{(1-\varepsilon)(1-\gamma+\alpha\omega+r\omega)-\theta(1-s)[1+\omega(1+r)]\}}{\varepsilon(1-\tau)(1-\gamma+\alpha\omega+r\omega)+[1+\omega(1+r)]\theta s\tau}.$$

*Then, the following generically holds:*

*(i) If $0 \le \varepsilon < \tau$ the steady state is always a saddle.*
*(ii) If $\tau \le \varepsilon < \varepsilon_c$ the steady state is a saddle when $-1 < \phi < \phi_g$; undergoes a transcritical bifurcation for $\phi = \phi_g$, becoming a sink when $\phi_g < \phi < \phi_c$.*
*(iii) If $\varepsilon_c \le \varepsilon < 1$ the steady state is a saddle when $-1 < \phi < \phi_g$; undergoes a transcritical bifurcation for $\phi = \phi_g$, becomes a source for $\phi_g < \phi < \phi_h$. A Hopf bifurcation occurs at $\phi = \phi_h$, and the steady state is a sink when $\phi_h < \phi < \phi_c$.*

From Proposition 6 we see that local indeterminacy and bifurcations may occur in our debt-constrained economy. Local indeterminacy (sink) always requires a sufficiently high production externality, $\varepsilon > \tau$ and sufficiently procyclical tax rates $\phi > \max(\phi_g, \phi_h)$ Also, Hopf, and transcritical bifurcations may occur.

## 5.1 | The local indeterminacy mechanism

Our local indeterminacy mechanism is due to the existence of endogenous borrowing constraints that make debt a forward-looking variable. It works as follows. Suppose that, departing from the steady state, at time $t$ agents expect future government debt, $b_{t+2}$, to increase. This would lose the current government borrowing constraint and from (20) government debt issued in time $t$, $b_{t+1}$, increases. Since $b_{t+1}$ has increased, $\phi < \phi_c$ guarantees that $k_{t+1}$ increases as well. More precisely, from Equation (32) we have that $\tilde{k}_{t+1} = \varepsilon\theta\left(1 - \frac{\tau\phi}{1-\tau}\right)\tilde{b}_{t+1} = \varepsilon\theta\frac{\tau}{1-\tau}(\phi_c - \phi)\tilde{b}_{t+1}$. This in turn increases future income and future tax revenues. However, the increase in tax revenues is not enough to repay the additional debt issued. It follows that the government needs indeed to issue additional debt the next period, which confirms initial expectations.[34]

However, we still have to guarantee that the economy will return to the steady state in the absence of further shocks on expectations. From our previous analysis we know that this future reversal of the trajectory of debt requires a sufficiently large externality $\varepsilon$. Indeed, when $\varepsilon$ is small, namely, $\varepsilon < \tau$, the joint dynamics of capital and debt will be mainly determined by the evolution of debt, since capital accumulation response to the initial change in debt is very small. See Equation (32). Therefore, from Equation (33), the path of debt will be explosive, as (i) the negative effect of capital accumulation on debt, which depends on $\varepsilon$, is not sufficiently strong and (ii) with small values of $\varepsilon$ the coefficient of lagged debt is higher than one. So in our model, as in many others, sufficiently strong productive externalities make the emergence of indeterminacy more likely.

However, in contrast with most of the previous results on the effects of taxation on local stability, we find that local indeterminacy requires a sufficiently positive elasticity of the tax rate function with respect to income, that is, local indeterminacy is more likely under procyclical tax rates. Indeed, procyclical taxes facilitate the reversal of the debt trajectory: tax revenues need eventually to become large enough so that debt can decrease in the future. This is possible only for a sufficiently procyclical tax rate.

In a closed economy framework with no debt, Chen and Guo (2013) also find that tax rate rules that are sufficiently increasing in income generate local indeterminacy. In their model, like in ours, the strength of productive externalities is linked to the ability of the government to raise income. In their case, the government runs a balanced budget, so that government spending externalities are obviously higher with progressive tax rates.[35] In our case, a procyclical tax rate rule generates higher tax revenues for the same level of debt and income. From (8), this allows the government to increase public spending, since the investment in production-enhancing activities is pinned down by the binding conditionality constraint (9). A higher level of spending, will lose the no-default constraint (10), which in turn allows the government to borrow more in the international market, therefore increasing investment in activities that generate production externalities.

As noted in Section 1, our local indeterminacy mechanism, linked to the existence of endogenous borrowing constraints, does not rely on the choice between consumption and leisure (which is not present in our model) which is at the basis of most previous works without productive government spending where local indeterminacy requires countercyclical taxation. This makes our work more appropriate for developing countries, frequently debt constrained and with underdeveloped financial markets, where hand-to-mouth households are the rule.

[34]Analytically, replacing the increase in $k_{t+1}$ into Equation (33) we obtain $\tilde{b}_{t+2} = \Gamma\tilde{b}_{t+1}$, where
$\Gamma = \frac{\gamma + \omega(1-\alpha) + \left[1 - \varepsilon\theta\left(1 - \frac{\tau:\phi}{1-\tau}\right)s(1+\phi)\right](1-\gamma+\alpha\omega+r\omega)}{\gamma+\omega(1-\alpha)+\varepsilon(1+\phi)(1-\gamma+\alpha\omega+r\omega)} > 0$, so that $b_{t+2}$ unambiguously increases, fulfilling the original expectations for all the admissible parameter values. Indeed, both the numerator and the denominator of $\Gamma$ are always positive for $-1 < \phi < \phi_c$ and $0 < \varepsilon < 1$.

[35]Their indeterminacy mechanism, however, operates through the choice between consumption and leisure.

## 6 | GLOBAL DYNAMICS

We can write the dynamic systems (22) and (23) as follows:

$$h_1(k_{t+1}, k_t, b_{t+1}) = k_{t+1} - \beta\left[1 - \delta + \left[1 - \tau\left(A(\alpha b_{t+1})^{\varepsilon} k_t^{s}\right)^{\phi}\right] A(\alpha b_{t+1})^{\varepsilon} s k_t^{s-1}\right)\Big] k_t = 0, \tag{34}$$

$$h_2(k_{t+1}, b_{t+1}, b_{t+2}) = \omega\tau\left[A(\alpha b_{t+2})^{\varepsilon} k_{t+1}^{s}\right]^{(1+\phi)} + [\gamma + \omega(1-\alpha)]b_{t+2} - [1 + \omega(1+r)]b_{t+1} = 0. \tag{35}$$

Let $(k_t, b(k_t))$ be the locus of points in the plane $(k_t, b_{t+1})$ where $k_t = k_{t+1}$. From Equation (34), we have that $h_1(k_t, k_t, b(k_t)) = 0$, that is,

$$\left[1 - \tau\left(A(\alpha b(k_t))^{\varepsilon} k_t^{s}\right)^{\phi}\right] A(\alpha b(k_t))^{\varepsilon} s k_t^{s-1} = \frac{\theta}{\beta}. \tag{36}$$

Consider next the locus of points $(k_t, \hat{b}(k_t))$ such that $b_{t+1} = b_{t+2} = \hat{b}(k_t)$. From Equations (34) and (35), this condition requires $k_{t+1} = \hat{k}_{t+1}$ such that $h_1(\hat{k}_{t+1}, k_t, \hat{b}(k_t)) = 0$ and $h_2(\hat{k}_{t+1}, \hat{b}(k_t), \hat{b}(k_t)) = 0$, that is,

$$\hat{k}_{t+1} = \beta\left[1 - \delta + \left[1 - \tau\left(A(\alpha\hat{b}(k_t))^{\varepsilon} k_t^{s}\right)^{\phi}\right] A(\alpha\hat{b}(k_t))^{\varepsilon} s k_t^{s-1}\right] k_t, \tag{37}$$

$$\hat{b}(k_t) = \left[\frac{1 - \gamma + \omega\alpha + \omega r}{\omega\tau(A\alpha^{\varepsilon})^{1+\phi}}\right]^{\frac{1}{\varepsilon(1+\phi)-1}} \hat{k}_{t+1}^{-\frac{s(1+\phi)}{\varepsilon(1+\phi)-1}}. \tag{38}$$

By construction, the steady states $(k, b)$ of the dynamic systems (22) and (23) satisfy $b = b(k) = \hat{b}(k)$, that is, they correspond to the crossing of the curves $b(k_t) = \hat{b}(k_t)$. Thus, we have one or two steady states depending on if the curves $b(k_t)$ and $\hat{b}(k_t)$ cross once or twice.

In Figure 2a–d we present several examples of the $b(k_t)$, in red, and $\hat{b}(k_t)$, in blue, curves, for different values of $\phi$ and $\varepsilon$, as explained below. Remember that along the function $b(k_t)$ we have $k_t = k_{t+1}$. For $(k_t, b_{t+1})$ such that $b_{t+1} > b(k_t)$ ($b_{t+1} < b(k_t)$), we have $k_{t+1} > k_t$ ($k_{t+1} < k_t$), so in the region above (below) the $b(k)$ function capital increases (decreases), that is, the horizontal arrows point to the right (left). Similarly, for $(k_t, b_{t+1})$ such that $b_{t+1} > \hat{b}(k_t)$ ($b_{t+1} < \hat{b}(k_t)$), we have $b_{t+2} > b_{t+1}$ ($b_{t+2} < b_{t+1}$), so in the region above (below) the $\hat{b}(k_t)$ function debt increases (decreases), that is, the vertical arrows point up (down).

## 7 | POLICY DISCUSSION

In this section, we resort to numerical examples to discuss the role of fiscal policy on the local and global properties of the dynamic system. Specifically, we consider three levels for the elasticity $\varepsilon$: a low ($\varepsilon = 0.1$), a medium ($\varepsilon = 0.35$), and a high ($\varepsilon = 0.5$) value. For each of these values, we discuss the role of fiscal policy for local and global indeterminacy by letting $\phi$ vary from $-1$ to $\phi_c$.

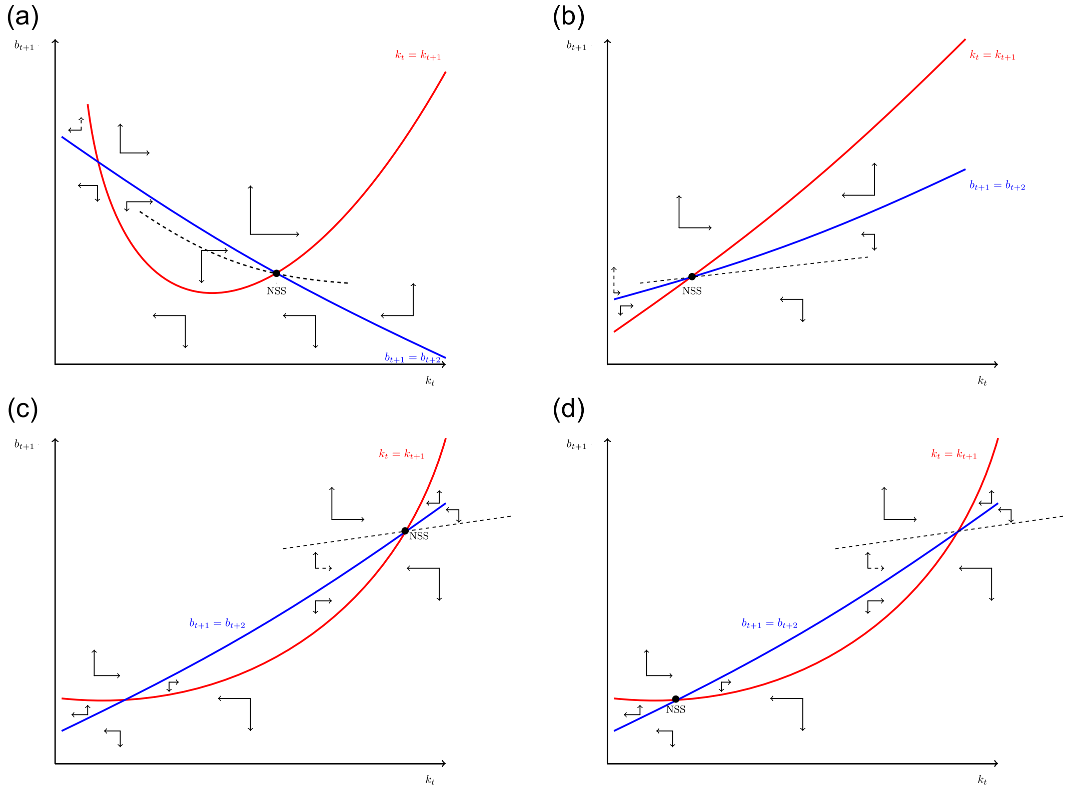


**FIGURE 2** The role of fiscal policy on global and local dynamics

When $\varepsilon = 0.1$ and $\phi$ is sufficiently negative, there exist two steady states, as in Figure 2a. The (normalized) high capital steady state is a saddle, whereas the low capital steady state is a source. Accordingly, all trajectories with an initial capital stock below the value corresponding to the source steady-state diverge. Also, any deterministic equilibrium trajectory with an initial capital stock above the value corresponding to the low steady state belongs to the stable manifold associated with the saddle point stable steady state.[36] As $\phi$ increases (eventually becoming positive), the global dynamics are as in Figure 2b, where the unique steady state is a saddle. These observations imply that for $\varepsilon$ small the economy displays neither local nor global indeterminacy.[37]

For $\varepsilon = 0.35$, if the tax rate is sufficiently countercyclical, the (normalized) high capital steady state is a saddle and coexists with a low capital steady state which is a source, as in Figure 2a. For mildly countercyclical and/or procyclical tax rates, the steady state is unique and a saddle, as in Figure 2b. When $\phi$ is sufficiently positive, the global dynamics become as in Figure 2c, where there exist two steady states: the normalized steady state is a saddle and the second steady state is a sink. Therefore, in this last configuration, deterministic trajectories that start on the saddle-stable manifold converge to the saddle steady state, whereas all

[36]Note also that, as the basin of attraction of the saddle-point stable steady state is bounded by the two isoclines, the source steady state may be connected to the saddle-stable steady state by a heteroclinic orbit. This can be proved by showing that the area between the two isoclines is a trapping area. For a detailed discussion see Brito et al. (2013).
[37]See Raurich (2000) for a definition.

deterministic trajectories that start below it converge to the low capital steady state,[38] which is a sink.[39] We can therefore say that the indeterminate sink steady state exerts an attraction of a global nature. Moreover, for a given value of the predetermined variable (capital), there are several different equilibrium trajectories that converge to different steady states. The observed deterministic equilibrium trajectory depends on the chosen value of the non-predetermined variable (debt).[40] This means that there is global indeterminacy, and that expectations about future debt determine not only the equilibrium trajectory, but also the long-run outcome of the economy.[41] Furthermore, in response to expectation shocks, the economy may switch between trajectories converging either to the low or to the high output steady state.

Finally, for sufficiently high values of $\varepsilon$, if the tax rate is sufficiently countercyclical the (normalized) high capital steady state is a saddle and coexists with a low steady state which is a source, as in Figure 2a. Again, as $\phi$ becomes larger, the steady state is unique and a saddle, as in Figure 2b. When the tax rate is mildly procyclical, there exists a second steady which is a sink, whereas the normalized steady state is still a saddle, as in Figure 2c. However, if $\phi$ is sufficiently positive, the normalized steady state becomes the low capital steady state and a sink, and the second steady state becomes a saddle (see Figure 2d). In the last two scenarios we have both local and global indeterminacy.

In summary, we conclude that, if the tax rate does not respond significantly to business cycles, the steady state is unique and saddle-stable, regardless of the size of $\varepsilon$. Moreover, when $\varepsilon$ is small, there is never global or local indeterminacy. In contrast, if $\varepsilon$ is sufficiently large and the tax rate is sufficiently procyclical, the low steady state is a sink and therefore an attractor of global nature.

From Equations (18), (19), and (25), steady-state government expenditures are increasing in steady-state output: $g = \frac{1-\gamma}{1-\gamma+\omega(\alpha+r)}\tau(y)y$. Since the government's objective is to maximize public spending, it will prefer unambiguously the high capital (high output) steady state. When $\phi$ is not too positive, the high output steady state is either unique and saddle-stable or it is saddle-stable and the only attractor equilibrium, as the lower steady state is a source. However, if the tax rate is sufficiently procyclical, provided $\varepsilon$ is not too small, the low output steady state becomes a sink and hence an attractor of a global nature. This implies that the government will prefer to avoid sufficiently procyclical tax rates to maximize long-run government spending. By doing this, the government will also prevent local or global fluctuations driven by volatile expectations.

This conclusion contrasts with the general wisdom that procyclical/progressive tax rates stabilize the economy in the presence of either exogenous shocks to fundamentals or expectations shocks.[42] As discussed in the previous section, in our setup, like in Chen and Guo (2013), tax rate rules that are sufficiently increasing in income generate local indeterminacy

[38]All the other deterministic trajectories diverge.

[39]Accordingly, there are infinitely many sunspots equilibria, that stay arbitrarily close to the sink steady state.

[40]Indeed, for a given $k_0$, for any choice of $b_1 < \varphi(k_0)$, where $b = \varphi(k)$ describes the stable manifold of the saddle, all equilibrium trajectories converge to the sink steady state, whereas when $b_1 = \varphi(k_0)$ the equilibrium trajectory converges instead to the saddle steady state.

[41]Global indeterminacy may also emerge under a balanced-buget rule in the presence of specific features of government spending. See, for example, Abad et al. (2020) with incompressible public expenditures in a Ramsey model, and Bella and Mattana (2019) with congestible public goods in a two-sector model.

[42]See Kletzer (2006) and Moldovan (2010) in the case of exogenous shocks to fundamentals and Guo and Lansing (1998) and Dromel and Pintus (2008) in the case of expectations shocks.

and therefore local expectation-driven fluctuations. Additionally, in our model procyclical tax rates introduce global indeterminacy and may generate larger endogenous fluctuations, where the economy switch between paths converging to different steady states in response to expectation shocks.

## 8 | CONCLUDING REMARKS

In this paper we study the interplay between government debt, fiscal policy, and macroeconomic instability. We consider a small open economy where the government maximizes spending to be reelected and, being unable to credibly commit to repay, faces endogenous borrowing constraints imposed by creditors, as well as a conditionality requirement designed to increase productivity. We show that the existence of endogenous borrowing constraints, making debt a forward-looking variable, may generate stochastic endogenous fluctuations driven by self-fulfilling volatile expectations (sunspots).

We also show that the choice of the tax rate rule critically determines steady-state properties and the dynamics of the economy. Indeed, steady-state multiplicity, which is necessary for the emergence of global indeterminacy, is more likely the greater the response of the tax rate rule to income is. Also, if the tax rate is sufficiently procyclical and productive externalities are not too small, the economy exhibits both local and global indeterminacy. This means that the economy will be subject to both local and global endogenous fluctuations driven by self-fulfilling volatile expectations. In contrast, if the response of the tax rate to the cycle is not too positive, regardless of the strength of the productive externality, local and global indeterminacy do not emerge, as the high output steady state is either unique and stable or it is saddle-stable and a global attractor. We conclude that by avoiding sufficiently procyclical tax rate rules the government guarantees convergence to the high output steady state, stabilizing the economy against local or global instability due to volatile expectations. Note that the government always prefers the steady state associated with the highest level of output, which also corresponds to higher levels of government spending. Therefore it follows that in the presence of borrowing constraints governments should not adopt tax rate rules that are too procyclical to maximize spending and stabilize the economy.

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## ORCID

*Francesco Carli* http://orcid.org/0000-0001-9594-4706
*Leonor Modesto* http://orcid.org/0000-0002-8302-2613

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## APPENDIX A

### A.1 | Solving the problem of capitalists

The first-order conditions of problem (4) are

$$c_{t+1}^{C} = \beta[1 - \delta + (1 - \tau(y_t))\rho_{t+1}]c_t^{C}, \tag{A1}$$

$$k_{t+1}^{C} = [1 - \delta + (1 - \tau(y_t))\rho_t]k_t^{C} - c_t^{C}. \tag{A2}$$

Iterating forward Equation (A2) we obtain

$$c_t^{C} + \sum_{j=1}^{\infty}\left(\prod_{i=1}^{j}[1 - \delta + (1 - \tau(y_t))\rho_{t+i}]^{-1}\right)c_{t+j}^{C} = [1 - \delta + (1 - \tau(y_t))\rho_t]k_t^{C}. \tag{A3}$$

Equation (A1) can also be written as

$$c_{t+j}^{C} = \beta^{j}\left(\prod_{i=1}^{j}[1 - \delta + (1 - \tau(y_t))\rho_{t+i}]\right)c_t^{C}. \tag{A4}$$

Substituting now Equation (A4) in (A3) we obtain expression (5). Finally substituting (5) in (A2) we obtain Equation (6).

## A.2 | Alternative specification for the government's utility function

In this section of the appendix we characterize the solution to the problem when the utility function of the government is strictly concave. In this case, the government's problem described by Equations (7)–(10) becomes

$$V_t(k_t, b_t) = \max_{(g_t, b_{t+1}, i_t)} u(g_t) + \gamma V_{t+1}(k_{t+1}, b_{t+1}) \quad \text{s.t. } (8)\text{-}(10). \tag{A5}$$

Using the same specification for the deviation payoff, $\overline{V}_{t+1}(k_{t+1}, b_{t+1}) = (1-\chi)V_{t+1}(k_{t+1}, b_{t+1}) + \xi b_{t+1}$, constraint (10) can again be rewritten as (11). Denoting again by $\lambda_t$, the Lagrange multipliers associated with (8) the first-order condition (12) becomes

$$\lambda_t = u'(g_t) > 0,$$

whereas (13) and (14) still hold. From the envelope condition $\frac{\partial V_{t+1}(k_{t+1}, b_{t+1})}{\partial b_{t+1}} = -\lambda_{t+1}(1+r)$. Replacing this last expression together with (12) and (13) in (14) we get

$$\mu_t = \frac{(1-\alpha)u'(g_t) - \gamma(1+r)u'(g_{t+1})}{1+\omega(1+r)u'(g_{t+1})}. \tag{A6}$$

As in the linear utility case, one dynamic equation is given by Equation (15). This is the dynamic equation for capital that we obtain from the capitalist problem and the equilibrium in the capital market. The second dynamic equation comes from solving (A6) and (17)–(19). When the government's utility is concave, and the no-default constraint binds, from (17), $b_{t+1} = \omega \cdot V_{t+1}(k_{t+1}, b_{t+1})$ and $\mu_t > 0$. Using (A5), we can rewrite (11) as

$$\begin{aligned} b_{t+1} &= \omega \cdot V_{t+1}(k_{t+1}, b_{t+1}) = \omega[u(g_{t+1}) + \gamma V(k_{t+2}, b_{t+2})] \\ &= [\omega \cdot u(g_{t+1}) + \gamma b_{t+2}]. \end{aligned}$$

Substituting now the government budget constraint (18) and expression (19) in this last equation we obtain

$$\omega \cdot u[\tau(y_{t+1})y_{t+1} + (1-\alpha)b_{t+2} - (1+r)b_{t+1}] + \gamma b_{t+2} - b_{t+1} = 0, \tag{A7}$$

which is the second dynamic equation.

Substituting $y_{t+1}$ and $\rho_t$ with $l_t = 1$, respectively, in (15) and (A7), we obtain the following two-dimensional dynamic system:

$$k_{t+1} = \beta\left[1 - \delta + \left[1 - \tau\left(A(\alpha b_{t+1})^{\varepsilon} k_t^{s}\right)^{\phi}\right] A(\alpha b_{t+1})^{\varepsilon} s k_t^{s-1}\right] k_t, \tag{A8}$$

$$\omega \cdot u\left[\tau\left[A(\alpha b_{t+2})^{\varepsilon} k_{t+1}^{s}\right]^{1+\phi} + (1-\alpha)b_{t+2} - (1+r)b_{t+1}\right] + \gamma b_{t+2} - b_{t+1} = 0. \tag{A9}$$

Equation (A8) is identical to Equation (22). Equation (A9), although slightly more complicated, is similar to Equation (23), still involving $b_{t+2}$, $b_{t+1}$, and $k_{t+1}$. As in the case of a linear utility function, these two Equations (A8) and (A9), representing, respectively, capital accumulation and the evolution of sovereign debt, determine the dynamics of this economy through a two-dimensional dynamic system in $(k_t, b_{t+1})$, with only one predetermined variable, the capital stock. Debt issued in period $t$, $b_{t+1}$, on the contrary, is a forward-looking variable, only affected by expectations of future debt, $b_{t+2}$, and future taxes, and independent of the existing amount of debt at the beginning of the period as in the case of a linear utility function. We conclude therefore that our results do not hinge on the linearity of the government's utility function.

### A.3 | Alternative specification for the deviation payoff

In this section we clarify the role played by the specification chosen for the deviation payoff, $\overline{V}_{t+1}(k_{t+1}, b_{t+1})$. In particular, we show that the essential features of the analysis would not change if we had assumed that, upon default, the government is permanently excluded from credit markets.

The analysis of the model where, upon default, the government is excluded from credit markets, requires us first to characterize the ensuing deviation payoff upon default $\overline{V}_{t+1}(k_{t+1}, b_{t+1})$. Following the literature on sovereign debt, we assume that upon default the production function suffers a productivity loss. A similar assumption is made, for example, in Cole and Kehoe (2000), who also assume that the economy becomes less productive when it loses access to financial markets. In our model, to keep things simple, we can represent this feature by assuming that the production function upon default has no productive externality, which we normalize to 1. Thus, if we let $\bar{y}_{t+1}$ and $\bar{g}_{t+1}$ denote GDP and government expenditures at date $t+1$ after the government has defaulted, using the government budget constraint we can rewrite the deviation payoff as follows:

$$\bar{V}_{t+1}(k_{t+1}, b_{t+1}) = \bar{V}_{t+1}(k_{t+1}) = \sum_{s=t+1}^{\infty} \bar{g}_s = \sum_{s=t+1}^{\infty} \tau(\bar{y}_s)\bar{y}_s. \tag{A10}$$

The government's problem remains identical to the one in the main text, that is, is still given by Equations (7)–(10), but now with $\overline{V}_{t+1}(k_{t+1}, b_{t+1})$ given by (A10). The first-order conditions for optimality are the same as Equations (12)–(14) in the main text. From the envelope condition, we still obtain that $\partial V_{t+1}(k_{t+1}, b_{t+1})/\partial b_{t+1} = -\lambda_{t+1}(1+r)$, so that, as in the main text we have that $\mu_t = [(1-\alpha) - \gamma(1+r)]/(1+r)$. The first dynamic equation describing the evolution of the economy is identical to Equation (22) in the main text. The second dynamic equation comes from the evolution of bonds and it is the solution to (16)–(19). From (16), and assuming (as in the main text) that $0 < \gamma < \frac{1-\alpha}{1+r}$, it follows that $\mu_t > 0$. Hence, in (17) $V_{t+1}(k_{t+1}, b_{t+1}) = \overline{V}_{t+1}(k_{t+1})$, and $V_{t+2}(k_{t+2}, b_{t+2}) = \overline{V}_{t+2}(\bar{k}_{t+2})$ so that, using (7), we have

$$g_{t+1} + \gamma V_{t+2}(k_{t+1}, b_{t+1}) = \bar{g}_{t+1} + \gamma \bar{V}_{t+2}(\bar{k}_{t+2}), \tag{A11}$$

$$g_{t+1} = \bar{g}_{t+1} = \tau(\bar{y}_{t+1})\bar{y}_{t+1}. \tag{A12}$$

Solving (18) for $g_t$ and substituting in (A12) this value and the value for $i_t$ that we obtain from (19), we get

$$(1-\alpha)b_{t+2} + \tau(y_{t+1})y_{t+1} - \tau(\overline{y}_{t+1})\overline{y}_{t+1} - (1+r)b_{t+1} = 0.$$

Substituting now $y_{t+1} = A(\alpha b_{t+2})^{\varepsilon} k_{t+1}^{s}$ and $\overline{y}_{t+1} = A k_{t+1}^{s}$ in the previous expression we obtain

$$(1-\alpha)b_{t+2} + \tau\left[A(\alpha b_{t+2})^{\varepsilon}k_{t+1}^{s}\right]^{1+\phi} - \tau\left[Ak_{t+1}^{s}\right]^{1+\phi} - (1+r)b_{t+1} = 0.$$

This is the second dynamic equation that characterizes the evolution of the economy. Although this equation is not identical to Equation (23) in the main text, it is qualitatively the same. Thus, under the assumption that, upon default, the government is permanently excluded from international borrowing, the evolution of the economy is described by a system of two dynamic equations similar to (22) and (23) in the main text, so that the essential features of the analysis do not change.

### A.4 | A slack no-default constraint

When the borrowing constraint is not binding, Equation (16) requires that $\gamma = \frac{1-\alpha}{1+r}$. In that case the level of government spending $g_t$ is indeterminate, and for each admissible level of government spending $g_t$, the economy would be described by Equations (22) and (18) that we rewrite below

$$k_{t+1} = \beta\left[1-\delta+\left[1-\tau\left(A(\alpha b_{t+1})^{\varepsilon}k_t^{s}\right)^{\phi}\right]A(\alpha b_{t+1})^{\varepsilon}sk_t^{s-1}\right]k_t, \tag{A13}$$

$$(1-\alpha)b_{t+1} + \tau\left[A(\alpha b_{t+1})^{\varepsilon}k_t^{s}\right]^{1+\phi} - (1+r)b_t - g_t = 0. \tag{A14}$$

Therefore, given a sequence of government expenditures $\{g_t\}_{t=0}^{\infty}$, the evolution of the economy is given by (A13) and (A14), a system of two dynamic equations in $k_t$, $k_{t+1}$, $b_t$, and $b_{t+1}$. Thus, given $\{g_t\}_{t=0}^{\infty}$, the initial conditions $b_0$ and $k_0$ fully determine the evolution of the system, so that, in this case both capital and debt are predetermined variables.

### A.5 | The Jacobian matrix

Rewrite the linear systems (32) and (33), in matrix form

$$\underbrace{\begin{bmatrix}1 & 0\\ c & d\end{bmatrix}}_{J_1}\begin{bmatrix}\tilde{k}_{t+1}\\ \tilde{b}_{t+2}\end{bmatrix} = \underbrace{\begin{bmatrix}a & b\\ 0 & e\end{bmatrix}}_{J_0}\begin{bmatrix}\tilde{k}_{t}\\ \tilde{b}_{t+1}\end{bmatrix}.$$

The Jacobian matrix, $J$, is then

$$J = J_1^{-1}\cdot J_0 = \begin{bmatrix}1 & 0\\ -\dfrac{c}{d} & \dfrac{1}{d}\end{bmatrix}\begin{bmatrix}a & b\\ 0 & e\end{bmatrix} = \begin{bmatrix}a & b\\ -\dfrac{ca}{d} & \dfrac{e-cb}{d}\end{bmatrix}. \tag{A15}$$

### A.6 | Trace and determinant

The trace, $T$, and determinant, $D$, of matrix $J$, which correspond, respectively, to the sum and product of the two roots (eigenvalues) of the associated characteristic polynomial $Q(\lambda) \equiv \lambda^2 - \lambda T + D$, are given by

$$D = \frac{[1+\omega(1+r)][(1-\theta(1-s))(1-\tau)-\theta s\tau\phi]}{(1-\tau)[\gamma+\omega(1-\alpha)+\varepsilon(1+\phi)(1-\gamma+\alpha\omega+r\omega)]}, \tag{A16}$$

$$T = \frac{(1-\tau)\{[1-\theta(1-s)][\gamma+\omega(1-\alpha)]+[1+\omega(1+r)] + \varepsilon(1-\gamma+\alpha\omega+r\omega)(1-\theta)\}}{(1-\tau)[\gamma+\omega(1-\alpha)+\varepsilon(1+\phi)(1-\gamma+\alpha\omega+r\omega)]} + \frac{\phi[(1-\tau)(1-\theta)\varepsilon(1-\gamma+\alpha\omega+r\omega)-\theta s\tau(\gamma+\omega(1-\alpha))]}{(1-\tau)[\gamma+\omega(1-\alpha)+\varepsilon(1+\phi)(1-\gamma+\alpha\omega+r\omega)]}. \tag{A17}$$

### A.7 | Geometrical method

From (A17) and (A16) the locus of points $(T_\phi, D_\phi)$, describing the values of $T$ and $D$ as $\phi$ takes different values for fixed values of the other parameters, is defined through the following linear expression, the $\Delta$ line:

$$D = -\Phi + ST, \quad \text{where}$$

$$\Phi = \frac{\varepsilon(1-\gamma+\alpha\omega+r\omega)(1-\theta)[1+\omega(1+r)][s\theta+(1-\tau)(1-\theta)] + \theta s\tau[1+\omega(1+r)]^2}{\varepsilon(1-\gamma+\alpha\omega+r\omega)\{\theta s[\gamma+\omega(1-\alpha)]+(1-\tau)[1+\omega(1+r)]\} + \theta s\tau[\gamma+\omega(1-\alpha)]^2},$$

$$S = \frac{[1+\omega(1+r)]\{\theta s\tau[\gamma+\omega(1-\alpha)]+\varepsilon(1-\gamma+\alpha\omega+r\omega)[(1-\tau)(1-\theta)+\theta s]\}}{\varepsilon(1-\gamma+\alpha\omega+r\omega)\{\theta s[\gamma+\omega(1-\alpha)]+(1-\tau)[1+\omega(1+r)]\} + \theta s\tau[\gamma+\omega(1-\alpha)]^2}. \tag{A18}$$

It is easy to see that $S > 0$ is a decreasing function of $\varepsilon$. We also have that $S = 1$ for $\varepsilon = \varepsilon_a$ whose expression is given in Appendix A.8.

The $\Delta$ line (see Figure A1) crosses line $AB$ for $\phi = \phi_f$, crosses the line $D = 1$ for $\phi = \phi_h$ and crosses the $AC$ line for $\phi = \phi_g$, whose expressions are given below in Appendix A.8. Moreover $\varepsilon_c$, whose expression is also given in Appendix A.8, is the value of $\varepsilon$ such that $\phi_g = \phi_h$ and $\varepsilon_b$, whose expression is also given in Appendix A.8, is the value of $\varepsilon$ such that $\phi_g = \phi_f$. The $\Delta$ line crosses the line $D = 0$ for $\phi = \phi_0 > \phi_c$, that is, outside the admissible range of values for $\phi$. See Assumption 5. In fact Assumption 5 implies that we have a half $\Delta$ line, only defined for values of $\phi$ in the interval $-1 < \phi < \phi_c$.

Also, from (A16), under Assumption 5 we have that $D > 0$ is a decreasing function of $\phi$. Therefore, as $\phi$ increases from $-1$ to $\phi_c = \frac{1-\tau}{\tau}$, the $\Delta$ line points downwards.

When $\phi = -1$, the determinant and the trace take the values $D_{\phi=-1}$, $T_{\phi=-1}$.

$$D_{\phi=-1} = \frac{[1+\omega(1+r)][(1-\theta)(1-\tau)+\theta s]}{(1-\tau)[\gamma+\omega(1-\alpha)]}, \tag{A19}$$

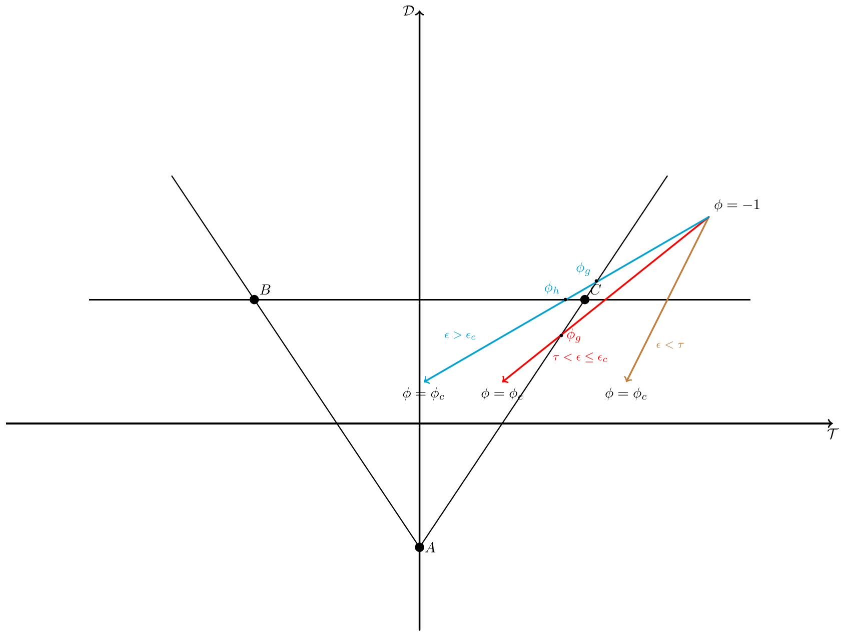


**FIGURE A1** Delta line

$$T_{\phi=-1} = \frac{(1-\tau)\{[1-\theta(1-s)][\gamma+\omega(1-\alpha)]+[1+\omega(1+r)]\}+\theta s\tau[\gamma+\omega(1-\alpha)]}{(1-\tau)[\gamma+\omega(1-\alpha)]}. \tag{A20}$$

Note that these values are not a function of $\varepsilon$. Therefore, for fixed values of the other parameters, instead of obtaining a line of initial points we just have an initial point $(T_{\phi=-1}, D_{\phi=-1})$ for all values of $\varepsilon$. It is straightforward to show that $D_{\phi=-1} > 1$ and that $D_{\phi=-1} < T_{\phi=-1} - 1$. We conclude that the half-line $\Delta$ always starts from the initial point $(T_{\phi=-1}, D_{\phi=-1})$ located below line $AC$ and above the line $D = 1$, pointing downwards as $\phi$ increases from $-1$ to $\phi_c$. As $\varepsilon$ increases from 0 to 1, the half-line $\Delta$ rotates to the left, its slope decreasing steadily from its maximum values $S_{(\varepsilon=0)} > 1$ and reaching the value $S = 1$ when $\varepsilon = \varepsilon_a$. For $\varepsilon > \varepsilon_a$, $S$ decreases from 1 to its minimum value $S_{(\varepsilon=1)} < 1$. See Figure A1.

From Assumption 5, we have the following ranking for the critical values of $\varepsilon$: $0 < \varepsilon_a < \varepsilon_b < \tau < \varepsilon_c < 1$.

When $0 < \varepsilon < \varepsilon_a$ the slope of the $\Delta$ half-line is higher than 1. This half-line starts for $\phi = -1$ below line $AC$ and above the line $D = 1$, pointing downwards as $\phi$ increases from $-1$ to $\phi_c$. As $\phi_0 > \phi_c$ the $\Delta$ half-line stops above the $D = 0$ line. Therefore the steady state is always a saddle.

When $\varepsilon_a < \varepsilon < \varepsilon_b$, where $\varepsilon_b$ is the value of $\varepsilon$ for which the $\Delta$ line crosses point $A$, the slope of the $\Delta$ half-line is smaller than 1, but it starts for $\phi = -1$ in the same initial point below line $AC$ and above the line $D = 1$, pointing downwards as $\phi$ increases from $-1$ to $\phi_c$. Again, as $\phi_0 > \phi_c$ it stops above the $D = 0$ line. Therefore the steady state is always a saddle.

When $\varepsilon_b < \varepsilon < \tau$, where $\tau$ is the value of $\varepsilon$ for which $\phi_g = \phi_c$, we have again $S < 1$, but now $\phi_g > \phi_h$. The $\Delta$ half-line crosses first the $BC$ line on the right of point $C$ for $\phi = \phi_h$, and later the $AC$ line below point $C$ for $\phi = \phi_g > \phi_c$. This means that the half-line $\Delta$ stops before crossing the $AC$ line. Therefore the steady state is again always a saddle. We conclude that we do not have indeterminacy when $0 < \varepsilon < \tau$. See the brown line in Figure A1.

When $\tau < \varepsilon < \varepsilon_c$, where $\varepsilon_c$ is the value of $\varepsilon$ for which the $\Delta$ line crosses point $C$, as in the case immediately above we have $S < 1$ and $\phi_g > \phi_h$. The half-line $\Delta$ crosses first the $BC$ line on the right of point $C$ for $\phi = \phi_h$, and later the $AC$ line below point $C$ for $\phi = \phi_g < \phi_c$. See the red line in Figure A1. It stops inside the triangle $ABC$ before crossing the $AB$ line. Indeed, it is easy to see that in this case the half-line $\Delta$ crosses the $AB$ line below the $D = 0$ line, that is, that $\phi_f > \phi_0 > \phi_c$. The steady state is a saddle for $-1 < \phi < \phi_g$ and a sink for $\phi_g < \phi < \phi_c$.

For $\varepsilon_c < \varepsilon < 1$, we have again $S < 1$, but now $\phi_g < \phi_h$. The half-line $\Delta$ crosses first the $AC$ line for $\phi = \phi_g$ and then the segment $BC$ in its interior for $\phi = \phi_h < \phi_c$. See the blue line in Figure A1. It stops inside the triangle $ABC$ before crossing the $AB$ line. Indeed the value $\varepsilon$ for which $\phi_f = \phi_c$ does not belong to the interval $[0, 1]$. Therefore the steady state is a saddle for $-1 < \phi < \phi_g$, a source for $\phi_g < \phi < \phi_h$ and then a sink for $\phi_h < \phi < \phi_c$.

All these results are summarized in Proposition 6 in the main text.

### A.8 | Expressions for critical values of $\phi$ and $\varepsilon$

$$\phi_c = \frac{1-\tau}{\tau},$$
$$\phi_g = \frac{(1-\tau)[\varepsilon-(1-s)]}{s\tau-\varepsilon(1-\tau)},$$
$$\phi_h = \frac{(1-\tau)\{(1-\varepsilon)(1-\gamma+\alpha\omega+r\omega)-\theta(1-s)[1+\omega(1+r)]\}}{\varepsilon(1-\tau)(1-\gamma+\alpha\omega+r\omega)+[1+\omega(1+r)]\theta s\tau},$$
$$\phi_f = \frac{(1-\tau)\{(2-\theta)\varepsilon(1-\gamma+\alpha\omega+r\omega)+[2-\theta(1-s)][1+\gamma+\omega(2+r-\alpha)]\}}{[1+\gamma+\omega(2+r-\alpha)]\theta s\tau-(2-\theta)\varepsilon(1-\tau)(1-\gamma+\alpha\omega+r\omega)},$$
$$\phi_0 = \frac{(1-\tau)[1-\theta(1-s)]}{\tau\theta s} > \phi_c,$$

$$\varepsilon_a = \frac{s\tau[\gamma+\omega(1-\alpha)]}{(1-\tau-s)[1+\omega(1+r)]+s[\gamma+\omega(1-\alpha)]},$$
$$\varepsilon_b = \frac{2s\tau[1+\gamma+\omega(2+r-\alpha)]}{(2-\theta)(1-\gamma+\alpha\omega+r\omega)(1-\tau-s)+[1+\gamma+\omega(2+r-\alpha)][(1-\tau)(2-\theta)+\theta s]},$$
$$\varepsilon_c = \frac{s\tau(1-\gamma+\alpha\omega+r\omega)}{s(1-\gamma+\alpha\omega+r\omega)-\theta[1+\omega(1+r)](1-s-\tau)},$$
$$\varepsilon_0 = \frac{s\tau[1+\theta(1-s)]}{1-\tau+\tau\theta s}.$$