Exogenous Spending in Models of Endogenous Growth

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Abstract

The literature of endogenous growth yields atypical solutions for optimal financing policies. In an environment with productive, exogenous spending, it is optimal to levy capital-income taxes, even if the government has access to non-distortionary instruments.

Furthermore, when government spending approaches its optimal value, the capital-income tax converges to zero. These results contradict the underlying assumption that the government's financing and spending decisions can be made independently of each other.

The literature cannot properly explain this result, nor does it grasp its economic significance. I show these results are not inherent to the models used to derive them. In fact, they stem from the assumption of exogenous spending, specifically how it is set.

The common framework in the endogenous growth literature is to set government spending as an exogenous share of output. When considering a reasonable deviation on this assumption, I show that non-distortionary taxes are indeed an optimal financing instrument. Thus, it is possible to implement optimal economic trajectories, while verifying the *separability assumption*.

´ **Key-words:** Exogenous spending; endogenous growth; separability assumption; optimal financing policy

Abstract

A literatura de crescimento endógeno obtém soluções atípicas como o governo deve financiar os gastos exógenos. Num ambiente onde os gastos do governo são exógenos e produtivos, taxar o capital é uma política ótima de financiamento, mesmo que seja possível taxar às famílias um valor fixo directamente (*lump-sum*).

Se os gastos público forem escolhidos optimamente, o imposto sobre o capital converge para zero. Estes resultados invalidam o pressuposto inicial de que as decisões sobre gastos e financiamento público podem ser feitas separadamente.

A literatura não consegue explicar a origem destes resultados, nem compreender a importância que os mesmos têm para o estudo de políticas ótimas de financiamento. Nesta tese é demonstrado que estes resultados não são inerentes aos modelos utilizados para os obter. São, na verdade, uma consequênica de considerar os gastos públicos como exógenos, ou melhor forma

Nesta literatura, é usual considerar que os gastos públicos correspondem a uma fração exógena de tudo o que é produzido na economia $G_t = gY_t$. Ao considerar outra forma de definir os gastos como exógenos, é possível mostrar que instrumentos não-distorcionários constituem políticas ótimas de financiamento público. Logo, é possível implementar trajectórias ótimas de crescimento sem violar a assunção sobre a separabilidade dos problemas do governo.

Palavras-chave: Gastos exógenos; crescimento endógeno; pressuposto de separabilidade; política ótima de financiamento

Contents

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1 Introduction

Barro (1990) puts forward a simple model of endogenous growth, in which government spending is productive. In this environment capital taxes are an optimal financing policy, even when there is access to lump-sum taxation. In fact, as public consumption approaches its optimal value, the capital-income tax converges to zero and the tax revenue must be generated lump-sum. These results go against the literature's benchmark, which dictates that exogenous spending is optimally financed by raising non-distortionary taxes.

Additionally, these results also contradict the assumption that government's spending and financing decisions can be made independently of each other. This *separability assumption* underlines the environment of the model. If unverified, the model becomes self-contradictory. Thus, it is important to understand whether these results are a consequence of the model's dynamics, or if they are driven by the assumptions adopted.

In this thesis I revisit the Barro model of endogenous growth, reverting the results originally presented in Barro (1990). I show that these results are a consequence of solving a Ramsey problem that takes spending as an exogenous share of output. I argue that this methodology creates an artificial co-movement between aggregate output the level of aggregate expenditures. Such co-movement is (and should) not be observed by the individual optimizing agents. Since households and firms are small and atomistic, they act as if their decisions do not affect the aggregate allocations, specifically the level of government spending. Hence, the representative firm will always take the level of expenditures as exogenous, rather than the share.

My view is that this behaviour is not sub-optimal! In fact, this is a direct consequence of the assumptions made on the behaviour of individual, optimizing agents. Consequently, the solution to the second best equilibrium should mirror such behaviour, and not one arbitrarily defined by the economist. The former is exactly the standard approach on the Optimal Taxation literature. Accordingly, Ramsey problems are solved taking a sequence of exogenous expenditures $\{G_t\}$ as given. The latter, I argue, is the approach adopted in Barro (1990) when considering spending as an exogenous share of output.

Before delving any further, it is important to highlight what this article is and what it is not. My goal is to provide a *normative* analysis on how the assumption of exogenous spending should be treated in models of endogenous growth. I do not intend to describe how optimal financing policies differ across different forms and values of exogenous spending. In fact, my view is that such *positive* approach to the study of financing policies is erroneous. If optimal policies are contingent on how spending is set, the government decisions can no longer be considered independent of each other. In this case, government spending could no longer be set as an exogenous. This thesis proposes a two-step methodology for the study of endogenous growth in an environment with exogenous spending. This methodology was constructed as to verify the benchmark results on the public finance literature, specifically the *separability assumption*. As a result, the problem of choosing optimal financing policies ought to be solved prior to the study of long-run endogenous growth.

A standard result on the Public Finance literature dictates that it is optimal to finance some exogenous spending with non-distortionary tax policies. The underlying assumption is that government's decisions over spending and financing can be separated into two distinct and independent problems. This *separability assumption* is what allows economists to abstract from one of this decisions, taking it as exogenous in their models. Thus, there are two "symmetric" problems that can be solved: 1) solve for the the optimal spending policy when given some exogenous tax revenue; or 2) solve for the optimal tax policy that finances exogenous expenditures. The latter is the standard framework in the literature of optimal financing policies.

I share the view presented in (Teles, 2012) by arguing that the use of such assumption categorically constrains the problem to be, at best, a *second best*. When both spending and financing decisions are endogenous, the solution is indeed optimal. This is a trivial result. The solution to any optimization problem is only attained when all variables are optimally chosen. In this case, endogenous government spending should be finandce by non-distortionary taxes. From now on, this solution will be referred to as the *first best* equilibrium.

Nevertheless, the use such assumption holds its rationale. First, it allows the researcher to abstract from an array of modeling issues that arise when augmenting a model with public expenditures. In real economies, governments provide an array of goods and services to families, for example Education and National Defense. Part of this provision can be productive, such as Public Capital and Education; while another part could be seen as welfare-enhancing (public parks, cultural services, etc). However, such specifications would only be relevant when solving for the optimal government provision of such goods and services. In other words, when solving for the optimal spending, given some exogenous tax revenue, it is paramount that the researcher properly specifies which part of expenditures is useful and which part is productive. In fact, this is precisely the reverse of the problem we want to focus on. Hence, once spending is assumed to be exogenous, we may disregard any questions on how these goods and services may impact the fundamentals of the model.

Additionally, the *separability assumption* simplifies the conceptual framework of our model-economy. If decisions over spending and financing and spending are independent of each other, the economist can take one of them as exogenous while focusing his study on the other. Actually, this puts our model-economies much more in-line with real economies, since budgetary decisions are not usually entrusted to a single bureaucrat. For example, one can reasonably expect financing decisions to be carried out by the (public) finance minister, while decisions over spending may fall under the control of the minister of the economy.

If government spending is exogenous, there is no reason for its value to be optimal (only by chance). Consequently, choosing non-distortionary tax policies to finance some exogenous level of government expenditures will always yield a *second best* result. The usefulness of the *separability assumption* is not challenged, the opposite in fact! However, the use of such assumption should be innocuous to any analysis. Specifically, the results on optimal financing policies should be consistent across different forms of exogenous spending. Additionally, the *first* and *second best* solutions should both be implemented by levying non-distortionary taxes. Thus, the two solutions should share same *principle* of taxation - to not distort the individual agents' decisions.

The standard assumption is to set government spending as an exogenous level -G_t. This is the approach shared by Ramsey (1927), Mirrlees (1971) and many others when studying optimal financing policies. A less common assumption, adopted by Lucas (2000) and Yun (2005), is to set government spending as an exogenous share -g_t- of some endogenous allocation (for example output Y_t, $G_t = g_t Y_t$). The assumption of an exogenous spending share is a convenient approach for the study of economic growth, since it allows the researcher to define the economy's trajectories with some mathematical ease. Barro (1990)and Turnovsky (1999), (2000) adopt this this approach.

In fact, the two assumptions share similar results as far as growth rates go. On one hand, we can consider an exogenous level of spending - G_t - to grow at the same rate as output. On the other, we can assume spending to be given as a constant, exogenous share of output - $g = \frac{G_t}{Y_t}$. Consequently, in both cases, the level of government spending would still grow with output [1]. Given this, the two assumptions may seem identical - both constrain the problem to be a second best, and both yield similar results regarding growth rates. Despite this parallel, the results for optimal financing policies are substantially different.

Under the assumption of an exogenous level of spending G_t, the *second best* solution is implemented by using non-distortionary taxes. Trivially, the *first best* solution would also be implemented under the same principle of taxation. If this parallel between *first* and *second best* financing policies holds to different forms of exogenous spending, then the *separability assumption* does not compromise the analysis. However, such parallel does not hold under the assumption of an exogenous spending share.

Teles (2012) delves into this issue. The author shows that under the assumption of an exogenous spending share the results for optimal financing policies are substantially altered. Using a model in which labour is the only input factor, the author shows that under the assumption of an exogenous spending share it is optimal to distort the margin between consumption and labour. Consequently, there would be welfare gains from raising consumption or labour-income taxes. In this framework, only using non-distortionary taxation would implements a third best solution.

Barro (1990) presents a model of endogenous growth in which government expenditures are productive. The environment is the one of a standard AK model [2]., in which production exhibits either constant or increasing returns to scale with respect to all inputs, but decreasing returns to each individual one. In his work, the author sets government spending as an exogenous share of output, which yields uncommon solutions for optimal financing policies.

On one hand, non-distortionary taxes are only optimal if government spending is endogenous. On the other, when spending is taken as exogenous it is optimal to levy taxes on capital. Then, in Barro's model the first and second best financing policies do not coincide in principle, violating the *separability assumption*. The author acknowledges there is a problem with this results, even though it fails to recognize its economic relevance:

"This result indicates that the income tax is not the only distortion in the model. I am

[1] This rationale holds whether growth is endogenous or exogenous to the model.

[2] The standard AK can be interpreted as the one-sector case of the model presented by Rebelo (1991)

uncertain whether the other distortion is economically interesting (...)" Barro (1990).

Barro argues justifies this results by arguing that the presence of exogenous spending creates an externality in the model. The literature followed suit by considering it as the correct framework for this class of endogenous growth models (Sala-i Martin (1990), Turnovsky (1999) and Liu and Turnovsky (2005)). But then, why was is this externality only mentioned when justifying the results of the model, instead of being explained since the outset? Moreover, what type of externality is this? Is it *Technological* or *Pecuniary*? And even more important, how can the existence of exogenous spending unilaterally create an externality in the model?

Broadly, an externality refers to the indirect effect created by the economic choices (consumption and productions) of a given agent. If such effect does not work through the price system, then the indirect effect is a *technological externality*, otherwise it is denoted as a *pecuniary externality* (Laffont, 1989). Technological externalities are, for example, the ones associated with pollution or with consumption-habit formation as modeled in Gali (1994) . In this type of externalities, private economic decisions have an effect on the economic choices of third parties.

On the other hand, a more recent literature studies optimal policy intervention in the presence of pecuniary externalities (Farhi and Werning (2016), Dávila and Korinek (2018) and Fanelli and Straub (2021)). This externalities arise when the individual decisions of price-taking agents have effects on the aggregate price level. A common view is that this type of externalities arise when the assumptions of the Arrow-Debreu theorem are relaxed. If this theorem holds, then prices only exist to clear the market. If instead there are frictions in the model, such as markets' incompleteness or asymmetric information, then the price has a different role, such as signalling the market. Relaxing the assumptions of the Arrow-Debreu theorem will yield sub-optimal effects on the equilibrium market prices, hence the market outcome can be improved through monetary policy interventions (Farhi and Werning (2016)).

Turnovsky (1999) and (2000) extend the Barro model to include an endogenous labour supply and investment-adjustment costs . The results are akin to those of the original setting - it is optimal to distort both the intra and inter-temporal margin. In fact, an optimal tax on capital became the benchmark result in the literature of endogenous growth in environments with productive government spending. It is then important to understand whether this results stem from the model itself or from the assumption of an exogenous spending share.

In this thesis, I show that in an environment similar to Barro (1990), it is possible to sustain balanced endogenous growth by only using lump-sum taxes. If spending is set as an exogenous level instead of an exogenous share, the first and second best economic trajectories are implemented under the same principle of taxation. During this thesis I present a systematic approach for the study of endogenous growth in environments with exogenous spending. This approach is such that the benchmark results on optimal financing policies are verified.

Furthermore, I show that the results of the Barro model are artificial, and stem purely from mis-using the assumption of an exogenous spendig share. Specifically, when solving for the second best solution under this assumption, the economist creates an artificial co-movement between the levels of spending and output. However, individual optimizing agents would not observe such co-movement upon making their economic decisions, as they can only observe the aggregate level of G.

Abusing the definition, a second best solution is a competitive equilibrium in which the financing policies are such that they maximize welfare. This conceptual equilibrium solution is obtained by solving an optimal-policy Ramsey problem. My view is that such problem should be solved under the assumption of an exogenous level of spending, as to mirror the behaviour of individual agents under a market outcome. Consequently, if the second best is obtained by setting spending as an exogenous share, the competitive equilibrium trajectories will not coincide with those of the second best solution. Hence, it would be optimal to levy distortionary taxes.

I am able to generalise this analysis to a model of endogenous growth with an endogenous labour supply, similar to the one in Turnovsky (1999) (2000) but without capital-adjustment costs. This is shown in sections (4) through (7), which also offers a comparison between the tax rules derived under different assumptions of government spending.

Finally, I use the results of this exercise to justify why one assumption should be used in lieu of the other. Hence, I am able to construct a normative criteria for how exogenous spending should be set. My stance is that the the assumption of an exogenous level of expenditures G_t is superior to the one of an exogenous spending share. This normative criteria

2 Literature Review

This thesis is anchored by two main branches of the economic literature: *Growth* and *Public Finance*. First, it delves into the economic growth literature by studying the assumption of exogenous spending in models of endogenous growth. This exercise reveals that the standard models fail to verify the benchmark results from the Public Finance literature. Teles (2012) is the the third cornerstone of this thesis, providing some intuition for these atypical outcomes.

2.1 Endogenous Growth

Initial models of economic growth, such as the Solow model, rely on exogenous technological or population changes to sustain long-run growth. During the 1980's the literature of endogenous growth emerged, providing an alternative to the well-established exogenous growth literature.

In this "new" class of models, long run growth is driven by the endogenous choices of individual optimizing agents. Romer (1986) and (1990) consider endogenous technological change as the source of long-run growth in a model in which *knowledge* is a productive input.[3] Alternatively, (Becker and Barro, 1988) study endogenous population changes as a driver of long-run growth.

A common "characteristic" of endogenous growth models is the presence of increasing or constant returns in the input factors that can be accumulated, such as physical capital (Rebelo (1991), Lucas Jr (1988)). In fact, this feature is a knife-edge assumption that must be adopted in order to sustain endogenous growth.

Barro (1990) extends the standard AK growth model to include productive government expenditures. [4] The economy is able to sustain endogenous growth even though capital exhibits diminishing marginal returns. Two critical assumptions underline this result: 1) government spending is also an accumulable factor; 2) aggregate production exhibits either constant or increasing returns to scale.

When augmenting a model with government spending, it is important to understand

[3] The author argues ideas/knowledge are, by nature, a non-rivalrous good, which implies the existence of increasing returns to scale. In this work private and social marginal returns will differ, leading to sub-optimal competitive equilibrium allocations.

[4] Barro's model can be seen as an AK model in which the aggregate accumulable factor K is disaggregated into both public and private capital.

what is included in this expenditures. In reality, the government provides different types of goods and services to the economy. Specifically, it is important to distinguish different provision with respect to their degree of rivalry and excludability. For a thorough analysis of this issue, and subsequent model extensions, I refer the reader to Barro (1990) and Sala-i Martin (1990).

Finally, Turnovsky studies different extensions of the Barro growth model, without ever addressing the issue of an exogenous spending share. For example, the author analyses this model in the presence of congestion costs (1995), or augments the initial set-up to include a labour market and investment-adjustment costs (1999).

2.2 Public Finance

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The Public Finance literature provides the normative criteria to the study of optimal financing policies. For example it states that government expenditures should be optimally financed with lump-sum taxes, regardless of spending being endogenous or exogenous to the model. This normative criteria relies on the assumption that government's spending and financing decisions can be solved independently of each other.

The literature of Optimal Taxation, pioneered by Ramsey (1927), has developed under this *separability* assumption. This branch of the (public finance) literature studies optimal financing policies in environments with restrictions on the use of lump-sum taxes. For a given sequence of exogenous spending, the Ramsey problem chooses the policies, from the set of feasible equilibrium solutions, that maximize the economy's welfare.

Another important result from the literature is the Ricardian Equivalence theorem. Barro (1974) [5] reopens a discussion on this topic by studying the (net) wealth effects from issuing bonds in different environments. This proposition states that, under certain conditions [6] , government's financing decisions have no effect on aggregate equilibrium allocations.

In fact, this proposition implies two different equivalences. On one hand, it implies that in any period t, taxes and bonds act as perfect substitutes in the financing of government

[5] Barro studies this topic apparently without mentioning past contributions Buchanan (1976).

[6] The conditions for the Ricardian Equivalence to hold can : 1)There is an infinitely lived representative agent; 2)Agents have access to perfectly competitive credit markets such that the interest rate on lending and borrowing is the same; 3) Taxes are lump-sum; 4) There are no market failures or frictions such as externalitie or markets incompleteness;

expenditures. For this intratemporal equivalence to hold, the government must be able to levy lump-sum taxes at least once, otherwise government debt would be rolled over forever. On the other, the proposition provides an equivalence between taxing in different periods. This intertemporal equivalence is usually framed as the timing of taxes being irrelevant.

Teles (2012) studies how the assumption of an exogenous spending share changes the benchmark result for Optimal Taxation policies. The environment is such that preferences over consumption and leisure are additively separable and output is produced by a constant-returns technology that only uses labour.

Broadly, the author shows that if spending is set an as an exogenous share of output it is optimal to distort the intra-temporal margin between consumption and leisure. The initial model is then extended to include preferences over public consumption. The assumption of an exogenous share also distorts the margin between public and private consumption. Thus, it would still be optimal to create a wedge in the consumption-leisure margin.

The author concludes that both assumptions affect the level of the tax rates, however, the assumption of an exogenous spending share also has implication on the cyclical properties of financing instruments. Finally, a more generic model is developed, which includes both uncertainty and contingent government debt. The results are akin to those of the simplest model studied.

3 An economy with productive government spending

This section presents a model economy similar to that of (Barro, 1990), but in which time is discrete. In this class of models, government spending is productive and taken as exogenous. As in the original model, I consider government expenditures to be a private good that is publicly provided to firms. Similarly, I assume that the amount of public spending G coincides with the total amount of productive goods and services provided by the government.

The economy is inhabited by a representative household, a representative firm and a government. In each period t the representative firm rents capital K_t and uses government spending G_t to produce a unique good. Output can be used for investment I_t, and both private C_t and public consumption G_t, such that

$$F(K_t; G_t) \geq C_t + G_t + I_t \tag{1}$$

Assuming that capital fully depreciates[7] at the end of each period, it follows that

$$I_t = K_{t+1}$$

F is a generic production function that exhibits either constant or increasing-returns to scale [8], and that satisfies the usual INADA conditions.

The representative household has preferences over consumption C described by the following lifetime utility

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{2}$$

In which u denotes the instantaneous utility of consumption.

In every period t, the government must finance some exogenous spending while running a balanced budget

$$G_t = \mathbf{T_t}$$

$\mathbf{T_t}$ is a policy vector that includes both distortionary and non-distortionary taxes.

[7] $\delta = 1$ provides a mathematical simplification to the model without carrying any loss of generality.

[8] To sustain endogenous growth, technology F must be such that it exhibits increasing or constant returns in the input factors that can be accumulated.

3.1 The Competitive Equilibrium

How Barro constructs the competitive equilibrium conditions is unclear. Sala-i Martin (1990) argues it is obtained by solving the problem of a representative household who owns the production of output [9]. Nevertheless, it can be obtained by solving the individual problems of each optimizing agent, whom have access to perfectly competitive markets.

Representative Household

In each period t, the representative household chooses consumption - C_t - and capital - K_{t+1} - to maximize (2) while satisfying the following intra-temporal budget constraint

$$C_t + K_{t+1} + T_t \leq u_t^K K_t(1 - \tau_t^K) + \pi_t \quad , \quad t \geq 0$$

Where u_t^K is the rental rate of capital and π_t are profits from the firm. T_t are lump-sum taxes, τ_t^K denotes the capital-income tax.

I assume U, defined in (2), to be strictly concave and twice differentiable, thus $u' > 0$ and $u'' < 0$ [10]. These assumptions ensure the solution to the Household problem is an interior optimum. Hence, it can be summarized in the following conditions

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = u_{t+1}^K(1 - \tau_{t+1}^K) \tag{3}$$

$$C_t + K_{t+1} + T_t = u_t^K K_t(1 - \tau_t^K) + \pi_t \tag{4}$$

Together with transversality condition

$$\lim_{T\to\infty} \frac{K_{T+1}}{\prod_{t=1}^{T} u_t^K(1 - \tau_t^K)} = 0 \tag{5}$$

Given $\left\{\pi_t; u_t^K; T_t; \tau_t^K\right\}_{t=0}^{\infty}$ and the initial level of the capital stock K_0, equations (3), (4) and (5) describe the solution of the representative household's problem for $\{C_t; K_{t+1}\}_{t=0}^{\infty}$.

[9] This framework is know in the literature as an household-producer problem.

[10] Let u' and u'' denote, respectively, the first and second derivative of the instantaneous utility with respect to consumption

Representative Firm

The representative firm takes the level of productive expenditures G_t as exogenous. Individual agents are small and atomistic and believe their optimal decisions will not affect the aggregate level of expenditures G_t. Therefore, even if government spending is set as an exogenous share g_t, the reasonable assumption is for the representative firm to take the level of spending G_t as exogenous.

In each period t, the representative firm maximizes profits

$$\pi_t = Y_t - u_t^K K_t \tag{6}$$

by choosing both output Y_t and capital K_t, subject to the Technological Constraint

$$Y_t = F(K_t; G_t) \tag{7}$$

and taking the exogenous sequence of government spending $\{G_t\}_{t=0}^{\infty}$ as exogenous.

The problem of the firm is further simplified as

$$\underset{\{K_t\}}{Max}\ \pi_t = F(K_t; G_t) - u_t^K K_t$$

The first order condition of this problem reads as follow

$$u_t^K = \frac{\partial F}{\partial K_t} \tag{8}$$

Given $\left\{G_t; u_t^K\right\}_{t=0}^{\infty}$, equations (7) and (8) define the representative firm's optimal choice for $\{Y_t; K_t\}_{t=0}^{\infty}$.

Government

In this economy the government has unlimited access to lump-sum taxes. Then, by the Riccardian Equivalence [11] , bond issuance is irrelevant. Therefore, it is possible to abstract from the cases in which the government issues debt. I will adopt this methodology for two reasons. First, it provides simplicity to the model. Second, this is the approach that

[11]This model satisfies the assumption that hold this result true. Even though there is no formal definition of the credit market, it can be seen as if it was endogenously cleared. The equilibrium would be such that household savings are zero and the real interest rate is equal to the return on capital net of taxes (no arbitrage).

appears to be adopted in Barro (1990).

Thus, the problem of the government is to choose in each period t the policies $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$, such that it verifies its budget constraint

$$G_t = T_t + \tau_t^K u_t^K K_t \tag{9}$$

Markets Clearing

In this economy there are only two markets, the output and capital assets market. The market for capital assets is endogenously cleared when solving the model.

In any period t, the goods market clearing condition can be written as

$$Y_t = C_t + G_t + K_{t+1} \tag{10}$$

The Competitive Equilibrium Solution

Definition 1 *Competitive Equilibrium*

In this economy, a competitive equilibrium is a set of allocations $\{C_t; K_{t+1}; Y_t; \pi_t\}_{t=0}^{\infty}$*, a price* $\left\{u_t^K\right\}_{t=0}^{\infty}$ *and policies* $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$*, such that, when given the initial capital stock* K_0 *and the exogenous sequence* $\{G_t\}_{t=0}^{\infty}$*, the following conditions are verified*

1. *Given* K_0*,* $\{\pi_t\}_{t=0}^{\infty}$*,* $\left\{u_t^K\right\}_{t=0}^{\infty}$ *and* $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$*, the representative household chooses* $\{C_t; N_t; K_{t+1}\}_{t=0}^{\infty}$ *to maximize (2) while subject to the sequence of intra-temporal budget constraints (4);*

2. *Given* $\left\{u_t^K\right\}_{t=0}^{\infty}$ *and* $\{G_t\}_{t=0}^{\infty}$*, the representative firm chooses* $\{Y_t; K_t\}_{t=0}^{\infty}$ *to maximize profits (6) subject to the technological constraint (7);*

3. *The government chooses* $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$ *while verifying its budget constraint (9);*

4. *Markets clear: condition (10) is satisfied;*

Equations (3) through (10) satisfy this definition [12]. However, by *Walras Law*, one of this conditions is redundant, and therefore it should be left out from the set of equilibrium conditions. A simple proof is provided below.

By substituting profits (6) in the Household budget constraint, the latter reads as

[12] Weak inequalities are satisfied with equality in equilibrium

$$C_t + K_{t+1} + T_t = u_t^K K_t(1 - \tau_t^K) + Y_t - u_t^K K_t$$

Substituting the markets clearing condition (10) in the equation above yields the government budget constraint

$$Y_t - G_t + T_t = -u_t^K K_t \tau_t^K + Y_t \leftrightarrow G_t = u_t^K K_t \tau_t^K + T_t$$

Thus, given K_0 and the exogenous sequence $\{G_t\}_{t=0}^{\infty}$, the set of competitive equilibrium solutions $\mathcal{E} \equiv \left\{C_t; K_{t+1}; Y_t; \pi_t; u_t^K; T_t; \tau_t^K\right\}_{t=0}^{\infty}$ can be described by the following conditions

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = u_{t+1}^K(1 - \tau_{t+1}^K)$$

$$C_t + K_{t+1} + T_t = u_t^K K_t(1 - \tau_t^K) + \pi_t$$

$$\pi_t = Y_t - u_t^K K_t$$

$$Y_t = F(K_t; G_t)$$

$$u_t^K = \frac{\partial F}{\partial K_t}$$

$$Y_t = C_t + G_t + K_{t+1}$$

Together with the transversality condition (5), which rules out solutions in which there is sub-optimal overaccumulation of capital.

The remaining equations are used to define the solution to the competitive equilibrium variables. In each period t, we have a system of 6 equations in 7 variables. Consequently, the competitive equilibrium solution is not unique. This is because the government has access to both lump-sum and capital-income taxes. For each unique policy vector $\mathbf{T}_{\mathbf{t}}^{*} = (T_t^*; \tau_t^{*;K})$ there is a unique, corresponding, competitive equilibrium solution.

The equations that define the equilibrium allocations $\{C_t; K_{t+1}; Y_t\}_{t=0}^{\infty}$ are sufficient conditions to characterize the entire competitive equilibrium solution. Given $\{K_t\}_{t=0}^{\infty}$, the rental rate of capital is defined by the firm's marginal condition (8). Then, and given $\{Y_t\}_{t=0}^{\infty}$, equation (6) defines profits. The policies $\{T_t; \tau_t^K\}$ are such that (4) is satisfied with equality.

Given $\left\{G_t; \tau_{t+1}^K\right\}_{t=0}^{\infty}$ and the initial capital K_0, the conditions that define the allocations $\{C_t; K_{t+1}; Y_t\}_{t=0}^{\infty}$ are as follows

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = (1 - \tau^K_{t+1})\frac{\partial F(K_{t+1};G_{t+1})}{\partial K_{t+1}} \tag{11}$$

$$Y_t = C_t + G_t + K_{t+1} \tag{12}$$

$$Y_t = F(K_t; G_t)$$

Together with

$$\lim_{T\to\infty} \beta^T K_{T+1} = 0 \tag{13}$$

Notice that the use of capital-income taxes distorts the capital accumulation margin, which leads to sub-optimal competitive equilibrium allocations.

3.2 Balanced endogenous growth

This section studies how the competitive equilibrium solution can be conditioned to move along a balanced growth path.

Assuming $F()$ is homogeneous [13] and exhibits constant returns to scale, the production function can be written as a standard *Cobb-Douglas*

$$Y_t = A_t K_t^{1-\alpha} G_t^{\alpha}$$

Assuming lifetime utility, defined in (2), is homothetic[14], the instantaneous utility function $u()$ can be represented by the isoelastic utility function.

$$u(C_t) = \begin{cases} \frac{C_t^{1-\sigma}-1}{1-\sigma} & \text{if} \quad \sigma \geq 0 \wedge \sigma \neq 1 \\ ln(C_t) & \text{if} \quad \sigma = 1 \end{cases}$$

In which σ denotes the inverse of the elasticity of substitution, constant in this case.

Considering the more generic case, lifetime utility would read as

[13] $H(x;y)$ is an homogeneous function of degree n if it can be expressed as $H(x;y) = x^n h\left(\frac{y}{x}\right)$.

[14] By the mathematical definition, a function is said to be homothethic if it can be written as a monotonic transformation of some other homogeneous function $g()$. The economic definition is stricter as it forces the degree of homogeneity of $g()$ to be one.

$$U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1 - \sigma}$$

Proposition 1 *Sustaining positive endogenous growth*

Given homothetic preferences and an homogeneous production function that exhibits either constant or increasing returns to scale in the factors that can be accumulated, the model is able to sustain positive endogenous growth, provided technology is sufficiently productive (A_t is sufficiently high).

The proof of Proposition 1 relies on U and F being homotheitc and homogeneous, respectively. If preferences are homothetic, the marginal rate of substitution between consumption periods t and $t+1$ can be written as a function of the ratio $\frac{c_{t+1}}{c_t}$. Consequently, the $MRS_{c_t;c_{t+1}}$ can be expressed as a function of the growth rate of consumption between two consecutive periods. If technology is constant returns to scale in the accumulable factors, then it can be written as an AK production function $Y_t = A_t K_t f\left(\frac{G_t}{K_t}\right)$. Consequently, the marginal productivity of capital will not necessarily decrease over time. Hence, the model will be able to generate positive endogenous growth. Considering the case of iso-elastic preferences, the Euler equation (11) reads as

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1 - \tau_{t+1}^K)(1 - \alpha) A_t \left(\frac{G_{t+1}}{K_{t+1}}\right)^{\alpha}$$

Given $C_{t+1} = (1 + \psi_t^C) C_t$, then

$$\left(1 + \psi_t^C\right)^{\sigma} = \beta(1 - \tau_{t+1}^K)(1 - \alpha) A_t \left(\frac{G_{t+1}}{K_{t+1}}\right)^{\alpha}$$

Hence, ψ_t^C can be positive if A_t is sufficiently large.

Consider now the set $\mathcal{G}$ which contains the infinite amount of sequences of government spending $\{G_t\}_{t=0}^{\infty}$. Consider the set $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ as well. This subset contains the sequences of government spending that can be expressed as a constant share of output $\{G_t = gY_t\}_{t=0}^{\infty}$. Trivially, any sequence of spending that belongs to $\tilde{\mathcal{G}}$ grows at the same rate as output.

Proposition 2 *Balanced Growth in period t*

If productivity is constant over time $A_t = A$ and the sequence of government expenditures is such that $\{G_t\}_{t=0}^{\infty} \in \tilde{\mathcal{G}}$, there is an endogenous growth path in which economic aggregates $\{Y; C; K\}$ grow at the same rate at any period t: $\psi_t^Y = \psi_t^C = \psi_t^K = \psi_t$

The proof of Proposition 2 is straightforward. If production exhibits constant returns to scale, and government expenditures grow with output, then the capital stock must grow with output as well. Given this, it follows from the resources constrain, described in (12), that the three economic aggregates must grow together. 2 implies that the ratio $\frac{G_t}{K_t}$ becomes constant over time and only depends on the initial exogenous K_0 and G_0. Hence, the growth rate of the economy at any period t can then be written as

$$1 + \psi_t = \left[\beta(1 - \tau_{t+1}^K)(1 - \alpha)A \left(\frac{G_0}{K_0}\right)^\alpha\right]^{\frac{1}{\sigma}}$$

Proposition 1 states the economy is able to sustain positive endogenous growth. Proposition 2 ensures that growth in the economy is balanced across economic aggregates $\{C; Y; K\}$, but not necessarily constant over time. In order to sustain constant balanced growth additional restrictions must be imposed on the equilibrium solution. Specifically, the government needs to commit to a time consistent policy τ^K in order for the competitive equilibrium solution to move along a balanced constant growth path.

Lemma 1 *Balanced Growth over time*

If productivity is constant over time $A_t = A$, $\{G_t\}_{t=0}^{\infty} \in \tilde{\mathcal{G}}$, and the government sets a constant capital income tax $\tau_t^K = \tau^K$, the economy is able to sustain balanced endogenous growth over time: $\psi_t = \psi$

In fact, from the moment Lemma 1 is verified, the economy is instantaneously along the balanced growth path.

Since expenditures in period $t = 0$ can be written as $G_0 = gY_0$, the competitive equilibrium balanced growth rate under a balanced growth path reads as

$$1 + \psi = \left[\beta(1 - \tau^K)(1 - \alpha)(Ag^\alpha)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} \tag{14}$$

Given the initial capital stock K_0, and the exogenous spending share g, the competitive equilibrium trajectories for $\{Y; K; C\}$ along the equilibrium balanced growth path are described by

$$K_t = K_0(1 + \psi)^t$$

$$Y_t = Y_0(1 + \psi)^t$$

$$C_t = C_0(1+\psi)^t$$

$$Y_0 = (Ag^{\alpha})^{\frac{1}{1-\alpha}} K_0$$

$$C_0 = Y_0(1-g) - (1+\psi)K_0$$

$$1+\psi = \left[\beta(1-\tau^K)(1-\alpha)(Ag^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

Together with the transversality condition defined in (13)

3.3 Second best equilibrium - Standard framework

The Ramsey problem is the literature's approach to the study of optimal financing tax policies. Different restrictions on this instruments result on different optimal tax rules.

A standard restriction is to set lump-sum taxes as negative $T_t < 0$. Instead of taxing, the government transfers lump-sum to the families. This environment, and comparable variations, are the common stance in the Optimal Taxation literature. Under this framework, the government's budget constraint is an active restriction on the competitive equilibrium of this economy. For the government to generate additional revenue, it needs to raise distortionary taxes (τ^K). Since each unique set of competitive equilibrium allocations $\{C_t; K_{t+1}; Y_t\}_{t=0}^{\infty}$ is contingent on a unique path of distortionary taxes $\left\{\tau_t^K\right\}_{t=0}^{\infty}$, changes in τ_k move the economy to a different competitive equilibrium solution. Hence, condition (9) binds the competitive equilibrium solution.

In a less interesting environment environment, the government has unlimited access to lump-sum taxes. Now, additional revenue can be always generated by taxing lump-sum. Thus, the government budget constraint does not bind the competitive equilibrium solution.

Definition 2 *Second Best Equilibrium*

In this economy, a second best equilibrium is a set of allocations $\{C_t; K_{t+1}; Y_t; \pi_t\}_{t=0}^{\infty}$, a price $\left\{u_t^K\right\}_{t=0}^{\infty}$ and policies $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$, such that, when given the initial capital stock K_0 and the exogenous sequence $\{G_t\}_{t=0}^{\infty}$, the following conditions are verified

1. *Given K_0, $\{\pi_t\}_{t=0}^{\infty}$, $\left\{u_t^K\right\}_{t=0}^{\infty}$ and $\left\{T_t; \tau_t^K\right\}_{t=0}^{\infty}$, the representative household chooses $\{C_t; N_t; K_{t+1}\}_{t=0}^{\infty}$ to maximize (2) while subject to the sequence of intra-temporal budget constraints (4);*

2. *Given* $\left\{u_t^K\right\}_{t=0}^{\infty}$ *and* $\{G_t\}_{t=0}^{\infty}$, *the representative firm chooses* $\{Y_t; K_t\}_{t=0}^{\infty}$ *to maximize profits (6) subject to the technological constraint (7);*

3. *The government budget constraint is verified (9);*

4. *Policies* $\left\{\mathcal{T} = T_t; \tau_t^K\right\}_{t=0}^{\infty}$ *maximizes (2);*

5. *Markets clear, conditions (10) is satisfied.*

The standard approach, to solve for the second best allocations, entails the choice of optimal financing policies for any sequence of exogenous spending $\{G_t\}_{t=0}^{\infty}$. If the government has unlimited access to lump-sum taxes, the second best can be always implemented as competitive equilibrium. In this case, the solution for the allocations can be obtained by maximizing 2, while subject to resources constraint 12.

The solution for the second best allocation is characterized by

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{\partial F(K_{t+1}; G_{t+1})}{\partial K_{t+1}} \tag{15}$$

$$Y_t = C_t + G_t + K_{t+1}$$

$$Y_t = F(K_t; G_t)$$

Implementability

To recover the second best policies, the second best solution must be discentralized to a competitive equilibrium. This is done by forcing the competitive equilibrium to verify the conditions for the second-best allocations.

Comparing condition (15) with the analogous competitive equilibrium condition (11), it follows that the government should not distort the intertemporal margin. Consequently, the government is able to implement the second best solution by only levying lump-sum taxes, which goes along with the benchmark of the Public Finance literature.

Thus τ^K is constant over time, which, by Lemma 1, ensures the second best equilibrium could be along a balanced growth path. If proposition 2 is verified, this equilibrium moves along a balanced growth path.

For the particular preferences and technology defined before, condition (15) reads as

$$(\frac{C_{t+1}}{C_t})^\sigma = \beta(1-\alpha)A_{t+1}(\frac{G_{t+1}}{K_{t+1}})^\alpha$$

Even though any sequence of government spending is able to sustain endogenous growth, balanced endogenous growth is only feasible if $\{G_t\}_{t=0}^\infty \in \tilde{\mathcal{G}}$. Considering that Lemma 1 is verified, the second best balanced growth rate would read as

$$1 + \psi_G = \left[\beta(1-\alpha)(Ag^\alpha)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} \tag{16}$$

Given the initial level of the capital stock K_0 and the exogenous spending share g, then trajectories of the economy can be descrived in

$$K_t = K_0(1+\psi_G)^t$$

$$Y_t = Y_0(1+\psi_G)^t$$

$$C_t = C_0(1+\psi_G)^t$$

$$(1+\psi_G) = \left[\beta(1-\alpha)(Ag^\alpha)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

$$Y_0 = (Ag^\alpha)^{\frac{1}{1-\alpha}}K_0$$

$$C_0 = Y_0(1-g) - (1+\psi_G)K_0$$

Together with the transversality condition.

Utility maximization and the optimal growth rate

In any period t along the balanced growth path, consumption reads as

$$C_t = C_0(1+\psi_G)^t$$

Given this, lifetime utility can be written as

$$U = \frac{C_0^{1-\sigma}}{1-\sigma}\sum_{t=0}^{\infty}\left[\beta(1+\psi_G)^{1-\sigma}\right]^t$$

Utility should be bounded, hence the growth rate ust verify

$$\beta(1+\psi_G)^{1-\sigma} < 1$$

$$U = \frac{C_0^{1-\sigma}}{1-\sigma} \times \frac{1}{1-\beta(1+\psi_G)^{1-\sigma}}$$

The maximum utility is attained by solving the following condition

$$\frac{\partial U}{\partial g} = 0 \leftrightarrow \frac{\partial U}{\partial C_0}\left[\frac{\partial C_0}{\partial g} + \frac{\partial C_0}{\partial \psi_G}\frac{\partial \psi_G}{\partial g}\right] + \frac{\partial U}{\partial \psi_G}\frac{\partial \psi_G}{\partial g} = 0$$

Which yields

$$g^* = \alpha$$

At any second best solution, the maximum of utility would be given by $g^* = \alpha$. This implies that the tax rate in (19) would converge to zero, which is a welfare-enhancing policy.

3.4 Barro's approach

In Barro (1990) it is not clear how the second best solution is obtained. In fact the model is solved as if the government only has access to an income tax, which in this economy will be equivalent [15] to a capital-income tax. Nevertheless, the author obtains the second best solution by setting government spending as a constant, exogenous share g of output. But how exactly can this solution be obtained and implemented if the government only has access to a distortionary tax? The implication made by Barro is that since the capital-income tax is constant, as to sustain balanced growth at the competitive equilibrium, the government spending share must be constant as well.

$$G_t = \tau Y_t \leftrightarrow \frac{G_t}{Y_t} = \tau = g$$

From which follows that the government exogenous spending share must be constant and equal to the tax rate $g = \tau$. In fact, when trying to implement a second-best solution "*à la Barro*"[16], the solution will differ from the standard one. This is because the competitive equilibrium solution in Barro (1990) is unique, hence the government budget

[15] Equivalent in the sense that it creates the same distortion. Both situation will distort the capital-accumulation margin, by creating an inter-temporal wedge.

[16] "*À la Barro*" in the sense that spending must be taken as an exogenous share due to restrictions on the government budget constraint. Section 5 further explores this issue.

constraint binds the solution for the tax.

Nevertheless, I obtain this second best solution, by employing t the methodoly described in section 3.3, but consider spending being set as an exogenous share of output.

In this case, the second-best solution for $\{C_t; K_{t+1}; Y_t\}$ is represented by

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = (1-g)\frac{\partial Y_{t+1}}{\partial K_{t+1}} \tag{17}$$

together with

$$Y_t(1-g) = C_t + K_{t+1}$$

and

$$Y_t = F(K_t; gY_t)$$

Comparing this solution with the competitive equilibrium conditions for $\{C_t; K_{t+1}; Y_t\}$, it follows from (11) and (17) that it could be optimal for the government to distort the inter-temporal margin. In this case the government would have to raise distortionary taxes on capital, and if its budget constraint does not verify, the government can always resort to lump-sum taxation. A question begs to be answered: *why is it optimal to create wedges on the margin between consumption through time?* A thorough explanation is provided below

The assumption of an exogenous spending share establishes a co-movement between aggregate output and the level of government spending. Since $G_t = gY_t$, increasing output by one unit leads to an increase of g units in government expenditures. This co-movement has two implications on the second-best solutions.

First, it distorts the production possibilities frontier for consumption in-between two consecutive periods. This production possibilities frontier can be obtained by combining the technology and resources constraint in periods $t+1$

$$C_{t+1} = (1-g)F(K_{t+1}; gY_{t+1}) - K_{t+2}$$

Together with the resources constraint in period t.

$$K_{t+1} = (1-g)Y_t - C_t$$

Which yields

$$PPF: \quad C_{t+1} = (1-g)F((1-g)Y_t - C_t; gY_{t+1}) - K_{t+2}$$

The condition above implies that the slope of the PPF is not only affected by spending being productive, but also by how the assumption of exogenous spending is set! Specifically, the slope of the PPF is distorted by the term $\frac{1-g}{1-\alpha}$.

$$\left.\frac{\partial C_{t+1}}{\partial C_t}\right|_{PPF} = -\frac{1-g}{1-\alpha}\frac{\partial F}{\partial K_{t+1}}$$

The intuition is as follows. The additional resources generated by forsaking one unit of consumption, and investing it in capital, are now used for both private and public consumption tomorrow. The government takes a share g of these additional resources to use as public consumption. The remaining part is used by the families to consume and invest in the following periods. Hence the marginal rate of technical substitution between consumption in t and $t+1$ (slope of the PPF) is distorted by the term $(1-g)$.

Secondly, since public consumption is a productive input, the assumption of an exogenous share has an additional effect on the marginal product of capital. The decision to invest one more unit of capital increases output by the amount of the marginal product of capital. As a result, the level of government spending $G_t = gY_t$ would expand by the proportion g. Then, and since spending is productive and $F()$ verifies the INADA conditions, the increase in G_t would raise the marginal product of capital in period t. For this reason the distortion of the PPF is dampen by the term $\frac{1}{1-\alpha}$.

Neither of this effect is observed by the representative firm when choosing capital and output. Consequently, the social and private marginal benefit of capital differs, leading to sub-optimal competitive equilibrium growth rates. This implies it can be optimal for the government to distort the equilibrium inter-temporal margin. [17]

[17]This is a direct application of the literature's *Second Best Principle*. For consistency, I shall denote this principle thesis as the *Third Best Principle*.

Implementability

Given proposition 2 is verified, this equilibrium is able to sustain balanced endogenous growth. Consequently, equation (17) yields the second-best growth rate under an exogenous spending share g, which reads as

$$1+\psi_g = \left[\beta(1-g)(Ag^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} \tag{18}$$

By comparing condition (14) with (18), it follows that the competitive equilibrium growth rate can only be second best if, and only if,

$$(1-g) = (1-\tau^K)(1-\alpha)$$

From which it follows that the second best solution is implementable by levying the following capital-income tax.

$$\tau^K = \frac{g-\alpha}{1-\alpha} \tag{19}$$

Under this tax-rule, condition 9 is only satisfied if lump-sum taxes are available. Under a balanced growth path, this condition reads as

$$G = gY = \tau^k u^K K + T$$

Given CRS and tje marginal condition of the firm,

$$gY = \tau^K(1-\alpha)Y + T$$

Given the optimal tax-rule for $t\tau^k$, it follows that lump sum taxes must be raised.

$$T = \alpha Y$$

This solution is quite atypical. Notice that lump-sum taxes are a residual policy, since it is only used when government spending is productive $\alpha > 0$. In fact this instrument is capturing the *pure economic rent* that exists in the economy. Notice that G is provided to the firms at no cost. If such expenditures had been transacted in competitive markets, the government would generate revenues the input factor G. Since technology is Cobb-Douglas, this revenue would be given by αY, with α denoting the share of input G in the production of output Y. Notice that since the government does not provide G through competitive markets, there are *economic rents* in the economy.

Hence, a possible intrepretation for the capital income tax is that it captures how the exogenous spending share deviates from its optimal value defined by $G = \alpha Y$. The larger the deviation, the greater the distortion. If the size of government - given by the exogenous share g - is too small, then the government should subsidize capital, otherwise the capital-income tax should be positive.

Economic trajectories under a balanced growth path

The optimal tax rule obtained in the previous subsection predicts a constant capital-income tax. Thus, lemma 1 is verified and the economy can sustain balanced endogenous growth.

Given the exogenous share g and initial capital stock K_0, the second-best equilibrium trajectories are described as follows.

$$K_t = K_0(1+\psi)^t$$

$$Y_t = Y_0(1+\psi)^t$$

$$C_t = C_0(1+\psi)^t$$

$$(1+\psi_g) = \left[\beta(1-g)(Ag^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

$$Y_0 = (Ag^{\alpha})^{\frac{1}{1-\alpha}} K_0$$

$$C_0 = Y_0(1-g) - (1+\psi_g)K_0$$

Together with the terminal condition (13).

Utility maximization and the optimal growth rate

It is trivial to understand that the maximum utility is only attained when both spending and financing decisions are endogenous to the model.

Under the assumption of an exogenous share, maximizing lifetime utility coincides with the problem of maximizing the growth rate (Barro, 1990). Hence,

$$\frac{\partial \psi_g}{\partial g} = 0 \leftrightarrow -(Ag^{\alpha})^{\frac{1}{1-\alpha}} + (1-g)\frac{\alpha}{1-\alpha}(Ag^{\alpha})^{\frac{1}{1-\alpha}}g^{-1} = 0$$

From which follows

$$\frac{1-g}{1-\alpha} \times \frac{\alpha}{g} = 1 \leftrightarrow \alpha = g$$

This is quite atypical. From the outset, the growth rate should be strictly increasing on government spending, regardless if the latter was set as a level or as a share. The larger the size of the government, the less resources are available for private consumption and investment goods. Consequently, the growth rate of economic outcomes between two consecutive periods would be quite large. The extreme case features output being completely used for government purchases. In this scenario, the solution for consumption is a corner solution $C = 0$ and the growth rate would be at its maximum. However, since consumption would be zero in all periods t, lifetime utility would not be at its maximum.

Therefore, maximizing lifetime utility should not correspond to the maximization of the growth rate. The fact that, in the Barro model, the two problems are coincident demonstrates how the assumption of an exogenous spending share hinders the analysis.

3.5 First Best Solution

The solution for the first best problem is in line with the results from section 3.4 and 3.3. This equilibrium can also be obtained by the solution to a Ramsey Problem, but in which both spending and financing decisions are chosen jointly.

If government expenditures are endogenous, then it is irrelevant whether it is the share or the level that is chosen. [18]

In this case the solution for $\{C_t; K_{t+1}; Y_t; G_t\}$ would read as

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{\partial F(K_{t+1}; G_{t+1})}{\partial K_{t+1}} \tag{20}$$

$$Y = C_t + G_t + K_{t+1}$$

[18] If spending is endogenous, using a spending share instead of a level is a simple change of variable.

$$Y_t = F(K_t; G_t)$$

$$\frac{\partial F}{\partial G_t} = 1$$

The first best equilibrium is a unique case of the second best solution, in which exogenous government spending is actually optimal. Therefore, the first best solution can be described by the conditions for a second best equilibrium together with an optimality condition for G_t. Given *Cobb-Douglas* technology, the optimal choice for G_t is such that in every period t the first order condition can be written as

$$\frac{\partial F}{\partial G_t} \times \frac{G_t}{Y_t} = \frac{G_t}{Y_t} \leftrightarrow \alpha = \frac{G_t}{Y_t} = g$$

Trivially, it coincides with the utility maximizing conditions presented in section **??**.

Sustaining balanced endogenous Growth

By Lemma 1 the first best solution is along a balanced growth path. Given the isoelastic utility and the aforementioned constant-returns technology, the first best growth rate reads as

$$1 + \psi_{FB} = \left[\beta(1-\alpha)(Ag^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

together with

$$g^* = \alpha$$

Hence,

$$1 + \psi_{FB} = \left[\beta(1-\alpha)(A\alpha^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

It follows that the competitive equilibrium is first best if and only if the capital-income tax is zero, and if spending is endogenous.

The trajectories of the first best solution are decribed by

$$K_t = K_0(1+\psi_{FB})^t$$

$$Y_t = Y_0(1+\psi_{FB})^t$$

$$C_t = C_0(1+\psi_{FB})^t$$

$$1+\psi_{FB} = \left[\beta(1-\alpha)(A\alpha^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

$$Y_0 = (z\alpha^{\alpha})^{\frac{1}{1-\alpha}}K_0$$

$$C_0 = Y_0(1-\alpha) - (1+\psi_{FB})K_0$$

Together with the transverlality condition and the initial stock of capital K_0

Comparison to the second best

Comparing the condition that defines ψ_{FB} with the condition for ψ_g and for ψ_G, it follows that the first-best allocations are always implemented by levying zero capital-income taxes and by using lump-sum taxation as a financing instrument.

On the other hand, when spending is set non-optimally the financing policies are contingent on the assumption of exogenous spending. Under the assumption of an exogenous spending share, the competitive equilibrium growth rate ψ_{CE} is second-best if the capital income tax is different from zero. However, if spending is set as an exogenous level the second best equilibrium is implemented by a lump-sum command, similar to the first-best implementation.

Consequently, first and second best financing policies only coincide in their principle of taxation when spending is set as an exogenous level. Hence, the assumption of an exogenous spending share cannot verify the *separability assumption*, which in turn would make it impossible to set spending as an exogenous.

4 An economy with productive government spending and an endogenous labour supply

This section extends the model-economy of section 3 to include an endogenous labour supply. In each period t the representative firm now hires labour N_t, rents capital K_t and uses government purchases G_t to produce an homogeneous good Y_t. Output is still used for investment I, and public G and private consumption C, such that

$$F(N_t; K_t; G_t) \geq C_t + G_t + I_t \tag{21}$$

with

$$I_t = K_{t+1} - (1 - \delta)K_t$$

Once again, I assume that capital fully depreciates at the end of each period t. Hence,

$$I_t = K_{t+1}$$

Let $F(.)$ still denote a generic production function that satisfies the usual INADA conditions.

The representative household now has preferences over consumption C and labour N, which can be described by the following lifetime utility

$$U = \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \tag{22}$$

In which $u(.)$ and $v(.)$ denote the instantaneous utility of consumption and the instantaneous disutility of labour, respectively.

The government still finance some exogenous spending while running a balanced budget

$$G_t = \vec{T}_t$$

The generic policy vector $\vec{T}_t$ now includes both capital and labour income taxes, as well as lump-sum taxes.

4.1 Representative Household

In each period t, the representative household, chooses consumption C_t, capital K_{t+1} and labour N_t to maximize (22) while satisfying the sequence of intra-temporal budget constraints

$$C_t + K_{t+1} + T_t \leq w_t N_t (1 - \tau_t^N) + u_t^K K_t (1 - \tau_t^K) + \pi_t, \quad t \geq 0 \tag{23}$$

Where w_t is the real wage and τ_t^N is the labour-income tax. All other have take on their usual meaning, which were described in section 3.1.

It is still the case that U, now defined by (22), must be strictly concave and twice differentiable, thus $u' > 0$, $u'' < 0$, $v' > 0$ and $v'' > 0$ [19]. These are sufficient conditions for the solution to be an interior optimum. Therefore, the following conditions summarize the solution of the Household problem

$$\frac{v'(N_t)}{u'(C_t)} = w_t (1 - \tau_t^N) \tag{24}$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = u_{t+1}^K (1 - \tau_{t+1}^K) \tag{25}$$

$$C_t + K_{t+1} + T_t = w_t N_t (1 - \tau_t^N) + u_t^K K_t (1 - \tau_t^K) + \pi_t$$

Together with terminal the condition for the accumulation of capital assets (13).

Given K_0, $\left\{\pi_t; w_t; u_t^K; T_t; \tau_t^K; \tau_t^N\right\}_{t=0}^{\infty}$ and the transversality condition, the three equations above depict the representative household's optimal choice for $\{C_t; N_t; K_{t+1}\}_{t=0}^{\infty}$.

4.2 Representative Firm

In each period t, the representative firm maximizes profits

$$\pi_t = Y_t - w_t N_t - u_t^K K_t \tag{26}$$

by choosing output Y_t, labour N_t and the capital stock K_t, while subject to the Technological Constraint

[19] Let v' and v'' denote, respectively, the first and second derivative of the instantaneous dis-utility of labour.

$$Y_t = F(N_t; K_t; G_t) \tag{27}$$

and to the exogenous sequence of government spending $\{G_t\}_{t=0}^{\infty}$.

In each period t, the problem of the firm can be further simplified as

$$\underset{\{N_t;K_t\}}{Max} \ \pi_t = F(N_t; K_t; G_t) - w_t N_t - u_t^K K_t$$

The firs order conditions read as follows

$$w_t = \frac{\partial F(N_t; K_t; G_t)}{\partial N_t} \tag{28}$$

$$u_t^K = \frac{\partial F(N_t; K_t; G_t)}{\partial K_t} \tag{29}$$

Given $\left\{G_t; w_t; u_t^K\right\}_{t=0}^{\infty}$, the solution for the firm's problem $\{Y_t; N_t; K_t; \pi_t\}_{t=0}^{\infty}$ is given by the conditions (26)-(29) together with the equation for profits.

4.3 Government

It is still the case that the government is forced to run a balanced budget.

Similarly to section 3.1, I assume the government is provided with enough policy instruments, such that *second best* solutions are always implementable. Lump-sum taxes should suffice, but I will allow the government to tax bot capital and labour income.

Thus, the problem of the government is to choose in each period t the policies $\left\{T_t; \tau_t^N; \tau_t^K\right\}$ such that it verifies the following budget constraint

$$G_t = T_t + \tau_t^N w_t N_t + \tau_t^K u_t^K K_t \tag{30}$$

4.4 Markets Clearing

The economy is composed of three markets: capital assets, output, and labour market. The markets for capital assets and labour are implicitly cleared when solving the competitive equilibrium. The goods market clearing condition still reads as

$$Y_t = C_t + G_t + K_{t+1} \tag{31}$$

4.5 Competitive Equilibrium

Definition 3 *Competitive Equilibrium In this economy, a competitive equilibrium is a set of allocations* $\{C_t; N_t; K_{t+1}; Y_t; \pi_t\}_{t=0}^{\infty}$*, prices* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and policies* $\left\{T_t; \tau_t^K, \tau_t^N\right\}_{t=0}^{\infty}$*, such that, when given the initial capital stock* K_0 *and the exogenous sequence* $\{G_t\}_{t=0}^{\infty}$*, the following conditions are satisfied*

1. *Given* $\{\pi_t\}_{t=0}^{\infty}$*,* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and* $\left\{T_t; \tau_{t+1}^K; \tau_t^N\right\}_{t=0}^{\infty}$*, the representative household chooses* $\{C_t; N_t; K_{t+1}\}_{t=0}^{\infty}$ *to maximize (22) subject to the sequence of intra-temporal budget constraints (23);*
2. *Given* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and the exogenous* $\{G_t\}_{t=0}^{\infty}$*, the representative firm chooses* $\{Y_t; K_t; N_t\}_{t=0}^{\infty}$ *to maximize profits (26) subject to the technological constraint (27);*
3. *The government chooses* $\left\{T_t; \tau_t^K; \tau_t^N\right\}_{t=0}^{\infty}$ *while satisfying its budget constraint (30);*
4. *Markets clear: condition (31) verifies;*

Given any sequence of government spending $\{G_t\}_{t=0}^{\infty}$ and a the initial capital stock K_0, any competitive equilibrium solution $\left\{C_t; N_t; K_{t+1}; Y_t; \pi_t; w_t; u_t^K; T_t; \tau_t^K, \tau_t^N\right\}_{t=0}^{\infty}$, can be described by the following conditions

$$\frac{v'(N_t)}{u'(C_t)} = w_t(1-\tau_t^N)$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = u_{t+1}^K(1-\tau_{t+1}^K)$$

$$C_t + K_{t+1} + T_t = w_t N_t(1-\tau_t^N) + u_t^K K_t(1-\tau_t^K) + \pi_t$$

$$Y_t = F(N_t; K_t; G_t)$$

$$\pi_t = Y_t - u_t^K K_t - w_t N_t$$

$$w_t = \frac{\partial F(N_t; K_t; G_t)}{\partial N_t}$$

$$u_t^K = \frac{\partial F(N_t; K_t; G_t)}{\partial K_t}$$

Together with the transversality condition (13).

It is still the case that this solution is not unique and that the equation (30) must be excluded from this set of conditions.

5 Balanced endogenous growth

The competitive equilibrium $\left\{C_t; N_t; K_{t+1}; Y_t; \pi_t; w_t; u_t^K; T_t; \tau_t^K, \tau_t^N\right\}_{t=0}^{\infty}$ can be fully characterized by the conditions that define equilibrium allocations. Given K_0 and $\left\{G_t; \tau_{t+1}^K, \tau_t^N\right\}_{t=0}^{\infty}$, such conditions read as follows

$$\frac{v'(N_t)}{u'(C_t)} = \frac{\partial F(N_t; K_t; G_t)}{\partial N_t}(1 - \tau_t^N) \tag{32}$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{\partial F(N_{t+1}; K_{t+1}; G_{t+1})}{\partial K_{t+1}}(1 - \tau_{t+1}^K) \tag{33}$$

$$F(N_t; K_t; G_t) = C_t + K_{t+1} + G_t$$

$$Y_t = F(N_t; K_t; G_t)$$

Together with the terminal condition (13)

Once again I shall assume lifetime utility defined in (22) is homothetic. Hence, it can be described by

$$U = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \theta \frac{N_t^{1+\eta}}{1+\eta}\right]$$

In which η denotes the inverse of the Frisch elasticity of labour supply, and θ is a preferences parameter that measures the distaste for labour.

Additionally, I will continue to assume a $Cobb - Douglas$ technology, but now it exhibits increasing returns to scale with respect to all inputs ($\alpha_N + \alpha_K + \alpha_G > 1$). In fact, the production function must still exhibit constant or increasing returns to scale in the factors that can accumulated ($\alpha_K + \alpha_G \geq 1$).

$$Y_t = A_t N_t^{\alpha_N} K_t^{\alpha_K} G_t^{\alpha_G}$$

Let α_i denote the share of input $i = \{N; K; G\}$ in the production of aggregate output Y. Let $\alpha_i \in (0; 1)$ for any $i \in \{N; K; G\}$, which implies decreasing returns to scale to all individual inputs.

Proposition 1 ensures this economy is able to sustain positive, unbalanced endogenous growth.

Consider once again the subset $\tilde{\mathcal{G}} \in \mathcal{G}$ presented in section 3.2.

Proposition 3 *Balanced growth at time t If productivity is constant over time $A_t = A$, $\{G_t\}_{t=0}^{\infty} \in \mathcal{G}$, and production exhibits constant returns to scale in the accumulable factors ($\alpha_K + \alpha_G = 1$), there is an endogenous growth path in which labour is constant and the economic aggregates $\{C; K; Y\}$ grow at the same rate.*

The proof of Lemma **??** follows straightforwardly from the technological and resources constraint. Since $\alpha_K + \alpha_G = 1$, we can define $\alpha_G = \alpha$ and (consequently) $\alpha_K = 1 - \alpha$, just as in the Barro model. If labour is constant, and spending grows at the same rate as output, then capital must grow at that same rate as well. From the resources constraint in (21) it follows that consumption must also grow with output.

Lemma 2 *Balanced growth over time Given that Proposition3 verifies, if the labour and capital-income taxes are constant over time, $\tau_t^N = \tau^N$ and $\tau_t^K = \tau^K$ respectively, the economy is able to sustain balanced endogenous growth.*

Once again, the moment these conditions are verified, the economy moves along the balanced growth path. The Euler equation in (33) would then read

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1-\tau^K)(1-\alpha)AN^{\alpha_N}\left(\frac{G_{t+1}}{K_{t+1}}\right)^{\alpha}$$

Since government expenditures grow with output, it is still the case that the ratio $\frac{G_t}{K_t}$ is constant over time. The competitive equilibrium growth rate under the balanced growth path can be written as

$$1 + \psi_{CE} = \left(\beta(1-\tau^K)(1-\alpha)AN^{\alpha_N}\left(\frac{G_{t+1}}{K_{t+1}}\right)^{\alpha}\right)^{\frac{1}{\sigma}}$$

Since technology is $Cobb - Douglas$ and government spending can be written as $G_t = gY_t$, then the growth rate is given by

$$1 + \psi_{CE} = \left(\beta(1-\tau^K)(1-\alpha)\left(AN^{\alpha_N}g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{\frac{1}{\sigma}} \tag{34}$$

Given K_0, g and the terminal condition (13), the economy's trajectories for $\{Y, C, K\}$ under the balanced growth path are decribed by

$$K_t = K_0(1+\psi)^t$$

$$Y_t = Y_0(1+\psi)^t$$

$$C_t = C_0(1+\psi)^t$$

$$Y_0 = (AN^{\alpha_N} g^{\alpha})^{\frac{1}{1-\alpha}} K_0$$

$$C_0 = Y_0(1-g) - (1+\psi)K_0$$

$$1+\psi_{CE} = \left(\beta(1-\tau^K)(1-\alpha)\left(AN^{\alpha_N} g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{\frac{1}{\sigma}}$$

Together with the condition for the intra-temporal margin between consumption and labour. This condition must verify that which labour N is constant along the balanced growth path.

$$N^{\eta} C_t^{\sigma} = (1-\tau^N)\alpha_N \frac{Y_t}{N} \tag{35}$$

It is easy to see that this condition holds when $\sigma = 1$. In fact, since ψ_{CE} is defined by labour N, it is possible for condition (35) to hold when $\sigma \neq 1$

6 Second best solution under the assumption of an exogenous level of spending G_t

Similar to section 3.3, this section studies optimal financing tax-policies by solving the standard Ramsey Problem.

Definition 4 *Second Best Equilibrium*

In this economy, a second best equilibrium is a set of allocations $\{C_t; N_t; K_{t+1}; Y_t; \pi_t\}_{t=0}^{\infty}$*, a price* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and policies* $\left\{T_t; \tau_t^N; \tau_t^K\right\}_{t=0}^{\infty}$*, such that, when given the initial capital stock* K_0 *and the exogenous sequence* $\{G_t\}_{t=0}^{\infty}$*, the following conditions are verified*

1. *Given* K_0*,* $\{\pi_t\}_{t=0}^{\infty}$*,* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and* $\left\{T_t; \tau_t^N; \tau_t^K\right\}_{t=0}^{\infty}$*, the representative household chooses* $\{C_t; N_t; K_{t+1}\}_{t=0}^{\infty}$ *to maximize (22) while subject to the sequence of intra-temporal budget constraints (23);*

2. *Given* $\left\{w_t; u_t^K\right\}_{t=0}^{\infty}$ *and* $\{G_t\}_{t=0}^{\infty}$*, the representative firm chooses* $\{Y_t; K_t; N_t\}_{t=0}^{\infty}$ *to maximize profits (26) subject to the technological constraint (27);*

3. *The government budget constraint is verified (30);*

4. *The policy vector* $\mathbf{T} = \left\{T_t; \tau_t^N; \tau_t^K\right\}_{t=0}^{\infty}$ *maximizes (22);*

5. *Markets clear, hence condition (31) holds*

In this environment, the solution for $\{C_t; K_{t+1}; Y_t; N_t\}$ is represented by

$$\frac{v'(N_t)}{u'(C_t)} = \frac{\partial F(N_t; K_t; G_t)}{\partial N_t} \tag{36}$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{\partial F(N_{t+1}; K_{t+1}; G_{t+1})}{\partial K_{t+1}} \tag{37}$$

$$Y_t = C_t + G_t + K_{t+1}$$

$$Y_t = F(N_t; K_t; G_t)$$

Together with the terminal condition for capital assets.

From the competitive equilibrium conditions (33) and (35), it follows that the government can implement the second best solution by not distorting the intra and inter-temporal margins. Hence, the hypothesis put forward is that optimalilty is attained by

$$\tau_{t+1}^K = 0$$

$$\tau_t^N = 0$$

Thus, it would be optimal for the government to finance exogenous expenditures by using non-distortionary policies. In this case, lump-sum taxes T_t and a tax on the income generated by the initial capital stock τ_0^K would be optimal.

Given Lemma 2, the growth rate under a balanced growth path is obtained from condition (37).

$$1 + \psi_G = \left[\beta(1-\alpha)\left(AN^{\alpha_N} g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} \tag{38}$$

Labour must be constant along this balanced growth path, which must be verified by equation (36). Therefore, along the balanced growth path this condition reads as

$$N^{\eta}C_t^{\sigma} = \alpha_N \frac{Y_t}{N} \tag{39}$$

Finally we can describe the economy's trajectories along the balanced growth path by conditions (38) and (39) together with

$$K_t = K_0(1+\psi)^t$$

$$Y_t = Y_0(1+\psi)^t$$

$$C_t = C_0(1+\psi)^t$$

$$Y_0 = (AN^{\alpha_N}g^{\alpha})^{\frac{1}{1-\alpha}}K_0$$

$$C_0 = Y_0(1-g) - (1+\psi_G)K_0$$

A comparison between the competitive equilibrium conditions (34) and (35) and those from the second-best solution ((38) and (39)) verifies the aforementioned intuition. Along this balanced growth path, capital and labour-income should go untaxed.

$$\begin{cases} (1-\tau^N)\alpha_N \frac{Y_t}{N} = \alpha_N \frac{Y_t}{N} \\ 1+\psi_{CE} = 1+\psi_G \end{cases}$$

The first condition defines the optimal tax on labour income $\tau^N = 0$. While the second one defines the optimal tax on capital $\tau^K = 0$.

$$\left[\beta(1-\tau^K)(1-\alpha)\left(AN^{\alpha_N}g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} = \left[\beta(1-\alpha)\left[AN^{\alpha_N}g^{\alpha}\right]^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

7 Second best solution under the assumption of an exogenous spending share

In this case, the *second best* solution is obtained by solving the Ramsey problem while taking an exogenous soending share as given. The solution can still be obtained by maximizing (22) subject to the resources constraint defined in (21) and to the exogenous spending share g

The solution to this problem for $\{C_t; N_t; K_{t+1}; Y_y\}$ is described by

$$\frac{v'(N_t)}{u'(C_t)} = (1-g)\frac{\partial F(N_t; K_t; gY_t)}{\partial N_t} \tag{40}$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = (1-g)\frac{\partial F(N_{t+1}K_{t+1}; gY_{t+1})}{\partial K_{t+1}} \tag{41}$$

$$Y_t(1-g) = C_t + K_{t+1}$$

$$Y_t = F(N_t; K_t; gY_t)$$

Once again, the assumption of an exogenous spending share creates a co-movement between output and the level of government spending. The distortion is now present in both the inter and intratemporal production margins. Hence, both the inter and intratemporal production possibilities frontier are distorted by the term $\frac{1-g}{1-\alpha}$. The reasoning presented in 3.4 is still valid, hence this second best solution should be implemented by levying both labour and capital income taxes[20]

Implementability and Economic Trajectories

The competitive equilibrium conditions (32) (33) imply that it might be optimal to distort both the consumption-leisure margin, as well as the margin between consumption across periods. Consequently it could be optimal for the government to raise distortionary taxes on labour and capital income, respectively. If the government budget constraint does not verify, the government can still budget its deficits by using non-distortionary instruments.

Given the conditions that verify Lemma 2, the intra and inter-temporal margins (40) and (41), respectively, read as follows

$$N^{\eta}C_t^{\sigma} = (1-g)\frac{\alpha_N}{1-\alpha}\frac{Y_t}{N} \tag{42}$$

[20] Third best principle application

$$1+\psi_g = \left[\beta(1-g)\left(AN^{\alpha_N}g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} \tag{43}$$

Once again, by comparing conditions (34) and (35) with the equilibrium conditions above, it follows that the competitive equilibrium can only be second best if, and only if,

$$\begin{cases} (1-\tau^N)\alpha_N \frac{Y_t}{N} = (1-g)\frac{\alpha_N}{1-\alpha}\frac{Y_t}{N} \\ 1+\psi_{CE} = 1+\psi_g \end{cases}$$

The first condition defines the optimal tax on labour-income

$$1-\tau^N = (1-g)\frac{1}{1-\alpha} \leftrightarrow \tau^N = \frac{g-\alpha}{1-\alpha}$$

While the second defines the optimal tax that should be levied on capital-income

$$\left[\beta(1-\tau^K)(1-\alpha)\left(AN^{\alpha_N}g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}} = \left[\beta(1-g)\left(AN^{\alpha_N}g^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

From which we obtain

$$(1-\tau^K)(1-\alpha) = (1-g) \leftrightarrow \tau^K = \frac{g-\alpha}{1-\alpha}$$

The conclusion that ensues is straightforward. It would be optimal for the government to levy *ad-valorem* taxes on both capital and labour-income. Specifically, the implication is that the government should raise an uniform distortionary tax on all different sources of income. Similarly to the ”optimal” capital tax deduced in section 3.3, this tax rates still capture how exogenous government spending deviates from its optimal value α. It is still the case that the greater the deviation, the greater the distortion. Moreover, the relation that was established between the size of the government and the sign of τ^i still verifies. If the government is too small (g is low enough), it is optimal for the government to subsidize both capital and labour, otherwise the tax rate is positive.

Lump-sum taxation still needs to be used whenever $\alpha > 0$. The lump-sum tax still captures the economic rents that exist in the economy. Now they are two-fold. On one hand there is still the pure rent generated from G not being transacted in perfectly competitive markets. On the other, since we have not imposed any conditions on α_N, it is possible that the firm has positive, zero or negative profits. This profits, generated from nonadjustable factors (i.e. decreasing returns to scale) constitute the second economic rent in this model, and should either be taxed lump-sum or by taxing profits. If there are increasing returns to scale, then it would be optimal to partially subsidize firms. Once again, I argue this result of lump-sum taxes acting as a residual policy that captures rents is a consequence of spending being productive and taken as an exogenous share.

8 First Best

The first best equilibrium can still be expressed as a second best equilibrium in which spending is endogenous. Thus we can describe this equilibrium by second best solution together with an optimality condition for the use of government expenditures.

Resource-wise, one more unit of expenditures carries a marginal cost of one unit. However, this additional expenditure is able to generate $\frac{\partial F}{\partial G}$ additional units of resources to the economy. Hence,

$$\frac{\partial F}{\partial G_t} = 1 \leftrightarrow \alpha A_t N_t^{\alpha_N} K_t^{1-\alpha} G_t^{\alpha} G_t^{-1} = 1 \leftrightarrow \alpha = \frac{G_t}{Y_t} = g$$

As technology is given by a *Cobb-Douglas* production function, the optimal choice of spending G_t^* is such that the spending to output ratio $\frac{G_t^*}{Y_t}$ is constant. Intuitively, the endogenous sequence of government spending $\{G_t^*\}_{t=0}^{\infty}$ belongs to the set $\tilde{\mathcal{G}}$, verifying proposition 3. Then, from (38), it follows that first best growth rate under a balanced growth path is given by

$$1 + \psi_{FB} = \left[\beta(1-\alpha)\left(AN^{\alpha_N}\alpha^{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

9 Concluding Remarks

A branch of the endogenous growth literature studies environment in which government expenditures are productive. The common framework in this literature is to set government spending as a a constant, exogenous share of output. Under this setting, optimal financing policies go against the economic literature benchmark. Specifically, this class of models violates both the *separability assumption* and the Chamley-Judd result.

In this thesis I revisit the Barro model of endogenous growth, a cornerstone model in such literature. In this class of modles, taxing capital is optimal, while lump-sum taxes may not even be needed to implement the second best solution.

By properly formalizing the optimal-policy Ramsey Problem, I am able to revert this results. If spending is taken as an exogenous share, there is an artificial co-movement created in the model. In this case, I prove that the second best solution is only implementable by distorting the capital accumulation margin. This contradicts the well-establish Chamley-Judd result, which dictates that capital should not be taxed in the long run.

Moreover, by comparing the first and second best solution, it follows that under the assumption of an exogenous spending share, the first and second best financing policies do not have the same principle of taxation. This implies that the separability assumption cannot be adopted invalidating the initial assumption that spending is endogenous.

I show that if spending is set as an exogenous level instead, the first and second best policies coincide in the principle of taxation - to not distort the equilibrium margins. Then, the Chamley-Judd result would be verified. This implies the assumption of an exogenous level is superior to assumption of an exogenous share, at least from normative perspective.

I prove that the Barro model is still able to sustain balanced growth. when spending is taken as an exogenous level. This is done by adopting a two-step methodology that requires solving the optimal financing problem prior to the study of endogenous growth.

Two criticism can be made to this analysis. On one hand, this thesis disregards some formalities under which the literature of endogenous growth is developed. For example, the use of continuous time is a standard practice in the study of economic growth. The literature also emphasises the study on the per-capita allocation growth. On the other, a critique can be made to how government spending is seen in this models. Government expenditures are treated as a flow/stream of goods, however, if spending is an accumulable factor, then it would be best describes as stock, such as human and physical capital.

I argue this drawbacks are irrelevant for my analysis. First, the primary goal of this thesis is to study how the assumption of an exogenous share hinders the results on optimal financing in models of endogenous growth. Specifically I want to analyse the effect of this assumption on optimal financing policies, rather than study growth by itself. Hence, adopting a proper and systematic approach to the study of a Ramsey problem in lieu of formality details is worthy trade-off.

Finally, I believe the second criticism is better directed to the endogenous growth literature as a whole, rather than to this specific work. Notice that the assumption of an exogenous spending share can still be adopted when spending is a considered a stock instead of a flow. Therefore, the second drawback is innocuous to the normative analysis developed in this thesis.

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A Annexes

A.1 The Barro Model

The Lagrangian of this household problem reads as follows

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(C_t) - \sum_{t=0}^{\infty} \lambda_t [C_t + K_{t+1} + T_t - u_t^K K_t(1-\tau_t^K) - \pi_t]$$

The First order conditions for each period t read as follows

$$(C_t): \quad \beta^t \frac{\partial u(C_t)}{\partial C_t} - \lambda_t = 0$$

$$(K_{t+1}): \quad -\lambda_t + \lambda_{t+1}(1-\tau_{t+1}^K) u_{t+1}^K = 0$$

$$(\lambda_t): \quad C_t + K_{t+1} + T_t - u_t^K K_t(1-\tau_t^K) - \pi_t = 0$$

The FOC for the multiplier (λ_t) yields the Household budget constraint (4). Condition (3) is obtained from the FOC for consumption, in period t and $t+1$, together with the one for capital in period t.

The transversality condition for this problem condition can be obtained from the FOC for capital in period $t = T$ with $T \to \infty$

$$(K_{T+1}): \quad \lim_{T\to\infty} K_{T+1} \frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0$$

Obtaining the Trasversality condition

$$\lim_{T\to\infty} K_{T+1} \frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \leftrightarrow \lim_{T\to\infty} K_{T+1} \lambda_T = 0$$

By recursively substituting the first order conditions for C_t and K_{t+1} in the equation above we obtain

$$\lim_{T\to\infty} \frac{K_{T+1} \frac{\partial u(C_0)}{\partial C_0}}{\prod_{t=1}^{T} u_t^K (1-\tau_t^K)} = 0$$

Given the solution is interior, then $\frac{\partial u(C_t)}{\partial C_t} > 0$ for any period t. Hence the transversality condition reads as

$$\lim_{T\to\infty} \frac{K_{T+1}}{\prod_{t=1}^{T} u_t^K (1-\tau_t^K)} = 0$$

Alternatively, we could have substituted the first order condition for consumption in period $t = T$. Then,

$$\lim_{T\to\infty} K_{T+1}\lambda_T = 0 \leftrightarrow \lim_{T\to\infty} K_{T+1} \beta^T \frac{\partial u(C_T)}{\partial C_T} \leftrightarrow \lim_{T\to\infty} \beta^T K_{T+1} = 0$$

Proof of Proposition 2

Proposition 2: *If productivity is constant over time $A_t = A$, the sequence of government expenditures is such that $\{G_t\} \in \tilde{\mathbf{G}}$, then economic aggregates $\{Y_t; C_t; K_{t+1}\}$ grow at the same rate at any period t: $\psi_t^C = \psi_t^Y = \psi_t^K = \psi_t$.*

Given $G_t = gY_t$ the technological constraint reads as

$$Y_t = AK_t^{1-\alpha}(gY_t)^\alpha$$

Which yields

$$Y_t^{1-\alpha} = AK_t^{1-\alpha}g^\alpha \leftrightarrow Y_t = (Ag^\alpha)^{\frac{1}{1-\alpha}}K_t$$

Assuming output that output can be written as $Y_{t+1} = (1+\psi_t)Y_t$

Then,

$$(Ag^\alpha)^{\frac{1}{1-\alpha}}K_{t+1} = (1+\psi_t)(Ag^\alpha)^{\frac{1}{1-\alpha}}K_t$$

Which yields

$$K_{t+1} = (1+\psi_t)K_t$$

The resources constraint in period t can be written as

$$C_t = Y_t - K_{t+1}$$

And in $t+1$ as

$$C_{t+1} = Y_{t+1} - K_{t+2}$$

Since capital and output grow at the same, then it follows that consumption must grow at the same as well.

$$C_{t+1} = (1+\psi_t)(Y_t - K_{t+1}) \leftrightarrow C_{t+1} = (1+\psi_t)C_t$$

Proof of Lemma 1

Lemma 1: *If productivity is constant over time $A_t = A$, $\{G_t\}_{t=0}^{\infty} \in \tilde{\mathbf{G}}$, and the government sets a constant capital income tax $\tau_{t+1}^K = \tau^K$, the economy is able to sustain balanced growth over time $\psi_t = \psi$.*

Given intertemporally homothetic preferences and a *Cobb-Douglas* production function, it follows

$$(\frac{C_{t+1}}{C_t})^\sigma = \beta(1-\tau^K)(1-\alpha)A(\frac{G_{t+1}}{K_{t+1}})^\alpha$$

If $\frac{G_t}{Y_t} = g$, then the ratio $\frac{G_t}{K_t}$ is also constant.

$$G_t = gY_t \leftrightarrow G_t = g(Ag^\alpha)^{\frac{1}{1-\alpha}}K_t \leftrightarrow \frac{G_t}{K_t} = g(Ag^\alpha)^{\frac{1}{1-\alpha}}$$

Then the Euler equation reads as

$$(1+\psi)^{\sigma} = \beta(1-\tau^K)(1-\alpha)A(g(Ag^{\alpha})^{\frac{1}{1-\alpha}})^{\alpha}$$

From which it follows that the growth rate under the balanced growth path is given by

$$1+\psi = \left[\beta(1-\tau^K)(1-\alpha)(Ag^{\alpha})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$

Second Best Solution: Exogenous Level of Spending

This Ramsey problems entails the maximization of utility subject only to the resources constraint. It is written as follows

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(C_t) - \sum_{t=0}^{\infty} \mu_t[C_t + K_{t+1} + G_t - F(K_t; G_t)]$$

$(C_t): \quad \beta^t \frac{\partial u(C_t)}{\partial C_t} - \mu_t = 0$

$(K_{t+1}): \quad -\mu_t + \mu_{t+1}\frac{\partial F}{\partial K_{t+1}} = 0$

$(\mu_t): \quad Y_t = C_t + K_{t+1} + G_t$

Second Best Solution: Barro's Approach

This Ramsey Problem maximizes utility subject both the resources and technological constraint. The Lagrangian to this problem is written as

The Lagrangian to this problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(C_t) - \sum_{t=0}^{\infty} \mu_t[C_t + K_{t+1} - (1-g)Y_t] - \sum_{t=0}^{\infty} \Omega_t[Y_t - F(K_t; gY_t)]$$

$(C_t): \quad \beta^t \frac{\partial u(C_t)}{\partial C_t} - \mu_t = 0$

$(K_{t+1}): \quad -\mu_t + \mu_{t+1}(1-g)\frac{\partial Y_{t+1}}{\partial K_{t+1}} -$

$-\Omega_{t+1}[\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \frac{\partial F(K_{t+1};gY_{t+1})}{\partial K_{t+1}} - \frac{\partial F(K_{t+1};gY_{t+1})}{\partial gY_{t+1}}\frac{\partial gY_{t+1}}{\partial K_{t+1}}] = 0$

$(Y_t:) \quad \mu_t(1-g) - \Omega_t[1 - \frac{\partial F(K_t;gY_t)}{\partial gY_t}\frac{\partial gY_t}{\partial Y_t}] = 0$

$(\mu_t): \quad (1-g)Y_t = C_t + K_{t+1}$

The FOC for output, Y_t can be simplified as

$$\mu_t(1-g) = \Omega_t[1 - \frac{\partial F(K_t; gY_t)}{\partial gY_t}\frac{\partial gY_t}{\partial Y_t}] \leftrightarrow \mu_t(1-g) = \Omega_t[1 - g\frac{\partial F(K_t; gY_t)}{\partial G_t}] \leftrightarrow$$

$$\leftrightarrow \mu_t(1-g) = \Omega_t[1 - \frac{G_t}{Y_t}\frac{\partial Y_t}{\partial G_t}] \leftrightarrow \mu_t(1-g) = \Omega_t(1-\alpha)$$

Given this, the FOC for capital K_{t+1} reads as

$$-\mu_t + \Omega_{t+1}(1-\alpha)\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \Omega_{t+1}[\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \frac{\partial F()}{\partial K_{t+1}} - \frac{\partial F()}{\partial gY_{t+1}}g\frac{\partial Y_{t+1}}{\partial K_{t+1}}] = 0 \leftrightarrow$$

$$\leftrightarrow -\mu_t + \Omega_{t+1}(1-\alpha)\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \Omega_{t+1}[\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \frac{\partial F()}{\partial K_{t+1}} - \frac{\partial F()}{\partial G_{t+1}}\frac{G_{t+1}}{Y_{t+1}}\frac{\partial Y_{t+1}}{\partial K_{t+1}}] = 0 \leftrightarrow$$

$$\leftrightarrow -\mu_t + \Omega_{t+1}(1-\alpha)\frac{\partial Y_{t+1}}{\partial K_{t+1}} - \Omega_{t+1}[\frac{\partial Y_{t+1}}{\partial K_{t+1}}(1-\alpha) - \frac{\partial F()}{\partial K_{t+1}}] = 0 \leftrightarrow$$

$$\leftrightarrow \mu_t = \Omega_{t+1}\frac{\partial F()}{\partial K_{t+1}}$$

Substituting the FOC for output in $t+1$ (Y_{t+1}), the condition reads as

$$\mu_t = \mu_{t+1}\frac{1-g}{1-\alpha}\frac{\partial F()}{\partial K_{t+1}}$$

Balanced growth path with an exogenous spending share

Given homothetic preferences and a *Cobb-Douglas* production function, second best solution is described by

$$\frac{C_{t+1}}{\beta C_{t+1}} = (1-g)\frac{\partial F(K_{t+1}; g_{t+1}Y_{t+1})}{\partial K_{t+1}}$$

$$Y_t(1-g) = C_t + K_{t+1}$$

$$Y_t = F(K_t; g_t Y_t)$$

A.2 A model with an endogenous labour supply

Household Problem

The Lagrangian of the representative household's problem presented in section 4 reads as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] - \sum_{t=0}^{\infty} \lambda_t [C_t + K_{t+1} + T_t - w_t N_t (1 - \tau_t^N) - u_t^K K_t (1 - \tau_t^K) - \pi_t]$$

In each period t the first order conditions for consumption, labour, capital and the multiplier, respectively, read as

$$(C_t): \quad \beta^t \frac{\partial u(C_t)}{\partial C_t} - \lambda_t = 0$$

$$(N_t): \quad -\beta^t \frac{\partial v(N_t)}{\partial N_t} + \lambda_t w_t (1 - \tau_t^N) = 0$$

$$(K_{t+1}): \quad -\lambda_t + \lambda_{t+1}(1 - \tau_{t+1}^K) u_{t+1}^K = 0$$

$$(\lambda_t): \quad C_t + K_{t+1} + T_t - u_t^K K_t (1 - \tau_t^K) - w_t N_t (1 - \tau_t^N) - \pi_t = 0$$

Equation (24) is obtained from the first order conditions for Consumption and Labour in period t. The first order conditions for consumption and the Lagrange multiplier yield condition (25).

Balanced Endogenous Growth

Proof of Proposition 3

Proposition 3: *If productivity is constant over time $A_t = A$, $\{G_t\} \in \tilde{\mathbf{G}}$ and production exhibits constant returns to scale in the accumulable factors ($\alpha_K + \alpha_G = 1$), there is an endogenous growth path in which labour is constant $N_t = N \forall t \geq 0$, and the economic aggregates $\{C; K; Y\}$ grow at the same rate.*

Given $\alpha_K + \alpha_G = 1$, then $\alpha_G = \alpha$ and $\alpha_K = 1 - \alpha$. Since $G_t = gY_t$ the technological constraint can be written as

$$Y_t = AN^{\alpha_N} K_t^{1-\alpha} (gY_t)^{\alpha}$$

Isolating output Y_t, it yields

$$Y_t^{1-\alpha} = AN^{\alpha_N} K_t^{1-\alpha} g^{\alpha} \leftrightarrow Y_t = (AN^{\alpha_N} g^{\alpha})^{\frac{1}{1-\alpha}} K_t$$

Assuming that output can be written as $Y_{t+1} = (1 + \psi_t) Y_t$. Then,

$$(AN^{\alpha_N} g^{\alpha})^{\frac{1}{1-\alpha}} K_{t+1} = (1 + \psi_t)(AN^{\alpha_N} g^{\alpha})^{\frac{1}{1-\alpha}} K_t$$

Which yields

$$K_{t+1} = (1 + \psi_t) K_t$$

The resources constraint in period t can be written as

$$C_t = (1 - g) Y_t - K_{t+1}$$

And in $t+1$ as

$$C_{t+1} = (1-g)Y_{t+1} - K_{t+2}$$

Since capital and output grow at the same, then it follows that consumption must grow at the same rate as well.

$$C_{t+1} = (1+\psi_t)((1-g)Y_t - K_{t+1}) \leftrightarrow C_{t+1} = (1+\psi_t)C_t$$

The equation for the intra-temporal margin between consumption and labour must then verify that labour is constant over time.

$$N^\eta C_t^\sigma = (1-\tau_t^N)\alpha_N \frac{Y_t}{N}$$

Notice that if Labour in not constant over time, output and capital would not grow at the same rate, and neither would consumption.

Let labour, capital and output grow at the rates $(1+\psi_t^N)$, $(1+\psi_t^K)$ and $(1+\psi_t^Y)$ respectively. Then, the technological constraint in period $t+1$ can be written as

$$(1+\psi_t^Y)Y_t = (AN_t^{\alpha_N} g^\alpha)^{\frac{1}{1-\alpha}}(1+\psi_t^N)^{\frac{\alpha_N}{1-\alpha}} K_t(1+\psi_t^K)$$

Since $Y_t = (AN_t^{\alpha_N} g^\alpha)^{\frac{1}{1-\alpha}} K_t$, it follows that

$$(1+\psi_t^Y) = (1+\psi_t^N)^{\frac{\alpha_N}{1-\alpha}}(1+\psi_t^K)$$

For output and capital to grow at the same rate $\psi_t^Y = \psi_t^K$, the following condition must be verified

$$1 = (1+\psi_t^N)^{\frac{\alpha_N}{1-\alpha}}$$

Given $\alpha \in (0;1)$, there are two possible solutions. Either Labour is not an input of production $\alpha_N = 0$, or Labour is productive $(\forall \alpha_N > 0)$ and constant $\psi_t^N = 0$ through time. Hence, Proposition 3 is satisfied.

Proof of Lemma 2

Lemma 2: *Given Proposition 2 holds true, if government sets constant capital and labour income taxes $\tau_t^K = \tau^K$ and $\tau_t^N = \tau^N$ the economy is able to sustain balanced growth $\psi_t = \psi$.*

Given intertemporally homothetic preferences and a *Cobb-Douglas* production function, it follows

$$(\frac{C_{t+1}}{C_t})^\sigma = \beta(1-\tau^K)(1-\alpha)\frac{Y_{t+1}}{K_{t+1}}$$

Given $Y_t = (Ag^\alpha N^{\alpha_N})^{\frac{1}{1-\alpha}} K_t$

Then the Euler equation reads as

$$1+\psi = \left[\beta(1-\tau^K)(1-\alpha)(Ag^\alpha N^{\alpha_N})^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\sigma}}$$