Estimation of the Market Price of Risk Implied in Stocks

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Palavras-Chave:

Risco, Determinação de Risco, preço de mercado do risco, Modelização de risco de crédito, S&P500, Ações, Valorização/Avaliação das ações, Modelos estruturais de análise de participações contingentes em empresas

Resumo:

Este trabalho implementa um modelo estrutural de valorização de participações em empresas. O modelo Silva (2017) baseia-se no modelo de Goldstein et al. (2001). No entanto, ao contrário deste último, considera-se a existência de custos fixos, para além de se escolher como variàvel de estado uma medida baseada em fluxos de caixa em vez do resultado operacional. O principal objectivo deste trabalho é calcular o preço do risco implicito no valor de mercado das ações. São utilizados dados relativos a empresas não financeiras pertencentes ao índice S&P 500 durante o período entre 1998 e 2017. As conclusões sugerem que existe um crescimento no preço de risco do mercado durante a bolha das "dot-com" e a crise financeira global de 2007-2008. Os resultados são, igualmente, comparados numa base sectorial, concluindo-se que nem todas as empresas e setores são igualmente afetados por estes grandes episódios económicos. Por último, o coeficiente de variação é calculado mostrando diferenças significativas ao nível do preço do risco entre empresas do mesmo setor. Adicionalmente, observou-se uma diminuição do coeficiente de variação do preço de mercado do risco, durante a crise financeira de 2007-2008 e a crise da dívida soberana sinalizando o caráter sistémico destas crises. O trabalho apresentado assenta no entanto em algumas hipoteses simplicadoras, como seja uma expectativa de crescimento constante e igual para todas as empresas em análise.

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Keywords:

Risk, Risk Pricing, Market Price of Risk, Credit Risk Modeling, S&P500, Equity, Equity Valuation, Structural Models of Corporate Contingent Claims.

Abstract:

This paper implements a structural model of corporate contingent claims. The model by Silva (2017) is based on the model of Goldstein et al. (2001). However, unlike the latter, fixed costs are taken into account. In addition the state variable is based on a cash flow measure instead of the operating results. The primary objective of this work is to compute the market price of risk implied in the market value of shares. Non-financial firms belonging to the S&P 500 index are used during the period 1998 to 2017. The findings suggest a that there is a growth in the market price of risk during the dot-com bubble and global financial crisis of 2007-2008. The results are also analyzed on a per sector basis, concluding that not all firms and sectors are equally affected by these major economic events. Lastly, the coefficient of variation was observed in the price of risk among companies in the same industry. In addition, the coefficient of variation was observed during the financial crisis of 2007-2008 and the sovereign debt crisis, indicating the systemic character of these crises. The work presented, however, is based on several simplified hypotheses, such as an expectation of constant and equal growth for all the companies under analysis.

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1 | Introduction

Equity valuation is the process of determining the fair value of a firm's equity. There are several important valuation methods in the equity valuation domain, these can be categorized into two groups: relative valuation and absolute valuation methods. These two methods differ significantly and choosing one of the methods often comes down to the availability of information. Price is the key metric for relative valuation methods, in which the focus is to calculate a ratio of the firm's share price to firm fundamentals. A comparison is then made to determine how a firm fares versus an industry benchmark. The benchmark is typically an average of several comparable firms in the same industry. Some example ratios are price-earnings (P/E) or price-sales (P/S). It is also possible, for example in a pre-revenue situation or in case financial data is sparse, to utilize non-financial multiples. These compare operating statistics such as website hits, unique visitors or subscriber counts with enterprise value. The relative valuation method is not typically used to calculate an exact dollar value per share, instead it is used to compare the current price to an underlying fundamental relative to other firms. On the other hand, absolute valuation methods attempt to calculate the firm's intrinsic value without taking into account its peers current market value. This method uses fundamentals such as cash flows and expected growth. A typical example is discounting all expected future cash flows (DCF).

These two methodologies for equity valuation are important tools in the toolkit of equity research teams and can provide solutions to several practical problems. Some of the common uses are issuing buy or sell recommendations, evaluating the impact of corporate events such as M&A or an LBO on firm value and establishing a fairness opinion in case of M&A transactions. However, equity valuation models are not restricted to stock picking. These models, along with most asset pricing models, can also be used to extract or infer market expectations on the model determinants, which is the goal of this thesis. The price of equity reflects not only the current performance but also the expectations of investors about the future performance of the firm. This price thus implies a certain set of assumptions adopted by the market regarding the firm's fundamentals. The analyst can evaluate these implied assumptions, compare them to his own expectations of the firm and then question how reasonable the market's assumptions are.

An example of an absolute valuation model is Gordon's dividend discount model, also referred to as the Gordon Growth Model, where expected dividends are discounted in order to obtain the price of the security today (Gordon and Shapiro, 1956):

$$Equity_0 = \frac{Div_1}{\mu_E - g}, \tag{1.1}$$

where Div_1 is expected dividends per share one year from now, μ_E is the discount rate and g is the dividend growth rate. In the case of this simple model it is relatively easy to reverse engineer specific outputs. For example, by assuming a value for μ_E, you can calculate the dividend growth rate implied in the equity valuation and by assuming a value for g, you can calculate the discount rate implied in the equity valuation. The work done for this thesis is similar to the previously mentioned example but uses a model that takes into account the shareholders' option to close the firm whenever it is optimal for them. Additionally, it uses free cash flow to the equity (FCFE) instead of dividends. The main reason for this is that many firms these days choose to substitute paying dividends with share repurchases, a phenomenon observed in both the U.S. (Grullon and Michaely, 2002) and European Union (von Eije and Megginson, 2008). Due to this shift, using dividends alone would be quite limiting. As FCFE is a measure of what the firm could theoretically return to the equity holders it bypasses the limitation of assuming only dividends.

This thesis implements a structural model of contingent liabilities that is used to compute an indicator of how risk is priced in the market at each moment in time. At first for individual firms and later the results are combined in a market cap weighted average. The results are also analyzed on a per sector basis. The calculations are done using a data set comprised of all non-financial firms listed in the S&P500 Index. A certain level of survivorship bias is introduced as only firms that are still included in the index at the time of the data collection are represented. The chosen model utilizes a cash flow-based state variable which is an alternative to EBIT or EBITDA based state variables. The main reason for using a cash flow-based state variable stems from the idea that cash is a more transparent measure of the financial state of a firm. As often said: 'Income is an opinion, cash flow is a fact'. Additionally, adopting a cash flow-based state variable means we can include several cash outflows which are not considered in an income statement. Capital expenditure (capex) being one of them. Capex is only taken into account indirectly on the income statement in the form of depreciation and amortization, yet ignores all cash outflows related to new investments. Another benefit is that the cash flow from operating activities (CFO) already takes into account corporate taxes, this means that the model only has to incorporate taxes on dividends.

The remainder of this thesis is structured as follows. The next section contains a literature review of contingent claim models. Chapter 2 further discusses the model used for this thesis. Chapter 3 describes the data set and how it was cleaned and filtered in preparation of running the model. Chapter 4 presents the empirical results of the model and the analysis of said results. Chapter 5 concludes this thesis.

1.1 Literature review

A contingent claim is any financial asset whose future payoff is dependent on the value of another asset, a derivative. Merton (1974) takes this concept and applies it to corporate debt. In this paper Merton described a firm with two classes of claims: debt in the form of a zero-coupon bond which makes up the main class and equity being the residual claim. At the time of maturity, T, the firm promises to pay the bond holders a predetermined amount B and if it fails to do so, assuming no distress costs, the bond holders can take over the firm while equity holders receive nothing. This means that at time T the firm has to pay the bond holders B or else face the equity becoming worthless, this leads to two scenarios. Scenario one: if the firm value is higher than amount B the firm should pay because the equity value is higher than zero. Scenario two: if the firm value is lower than amount B the firm would rather default on their debt as to avoid equity holders having to bring in more capital. In this scenario the debt holders receive the firm value at time T under the hypothesis that the firm value follows a geometric Brownian motion, these scenarios and payoffs can be thought of as a short put option on the assets of the firm with a strike set equal to the nominal value of the debt from the perspective of the debt holders. This means that the equity holder is the owner of the firm, has borrowed the amount B and is long the put option on the assets of the firm with strike equal to amount B. By virtue of the put-call parity relationship it is also possible to say that the debt holders own the firm and the equity holders instead have a call option on the assets of the firm at strike B, the nominal value of debt. The value of these options can be calculated using the formulas provided by Black and Scholes (1973). Merton (1974) then shows that the spread, which he refers to as a risk premium, between risky debt and otherwise similar but risk-free debt is the value of the aforementioned put option. Using this insight Merton is able to construct a risk structure of interest rates. An important note regarding Merton's model is that only moment T is important, everything before is irrelevant. This implies that, for example, extremely poor performance prior to time T has no effect on the model's output. It is not until several years later that an extension of Merton's model is presented that changes this.

That model is named a first passage time model and is introduced by Black and Cox (1976). In the first passage time model the events before time T can have a significant effect on its output, as hitting a certain threshold can trigger an event. Black and Cox justify this extension through several types of bond indenture provisions such as safety covenants, subordination arrangements and restrictions on the financing of interest and dividend payments. This extended model brings some additional changes. First of all, the zero-coupon debt is replaced by perpetual debt with a constant coupon rate. Secondly, by adding the safety covenants, Black and Cox adapt the unrealistic assumption of Merton's model where the firm can only default on its debt obligation at the time of maturity T.

The safety covenant gives creditors the right to force the firm into bankruptcy and take control of its assets as soon as the value of the firm falls below a certain threshold. This threshold is the default barrier ($\overline{v}$) and determines the maximum loss the debt holders can incur. When the firm value breaks through the barrier for the first time, the firm defaults giving $\overline{v}$ to the debt holders while equity holders receive nothing. Whereas in Merton (1974) the point of default was set, it is now uncertain. Next to uncertainty, the default barrier also leads to higher probabilities of default. This can be explained by noting that a number of paths in the Merton model that would previously not lead to default, now do lead to default under Black and Cox. In case the barrier equals nominal debt, there may be uncertainty regarding the time of default but there is no uncertainty regarding the amount recovered. The inclusion of a barrier implies that the equity now represents a down-and-out call option, which implies that a portion of the equity value is shifted to the debt holder as a result of the safety covenant. In the later part of Black and Cox (1976), it is shown that the default barrier can be endogenously determined by its shareholders through solving an optimal stopping time problem. In this situation, $\overline{v}$ is chosen by shareholders independently of the current value of the firm in order to minimize the value of the bonds and maximize the value of equity. In addition, the barrier is proportional to the debt repayments which implies that the higher the coupon payments, the higher the chance that the firm cannot repay the debt holders and thus the higher the value of $\overline{v}$. Lastly, the asset volatility level plays an important role in setting the barrier as shareholders are more willing to save the firm if there is a possibility that they will make large profits in the future. The model in Black and Cox (1976) does have some shortcomings. For example, it does not allow for debt that is coupon paying with a finite maturity, nor does it allow for the analysis of the optimal capital structure without introduction of taxes and default costs. These are some of the points that are improved upon in the model introduced by Leland (1994).

Leland (1994) builds on the earlier structural models of Merton (1974) and Black and Cox (1976) but incorporates taxes and distress costs. These allow the model to determine the optimal default boundary as well as closed form solutions for optimal capital structure. By incorporating taxes, the model can consider the tax benefits of leverage. Under the majority of taxation systems, firms are taxed on their profits. As such it can be favorable for the firm to leverage up, this decreases their accounting profits and lowers their taxes. Leland (1994) considers that these tax benefits resemble a security that pays a constant coupon equal to the tax-sheltering value of the interest payments for as long as the firm remains solvent. As distress costs lower the firm value available to debt holders, it can be interpreted as a claim on the assets of the firm whenever it defaults. This implies that debt has two counteracting effects: first it increases firm value due to the tax benefits and second, it reduces firm value due to bankruptcy costs. The model shows that firm value is calculated as the firm's assets value, plus the value of tax benefits, minus the distress costs. The value of equity is calculated by subtracting the value of debt from the total value of the firm. There are two possible triggers for default in this model, depending on whether the debt is protected or unprotected. In the case of unprotected debt, the default point is endogenously determined, the firm chooses the default barrier so as to maximize the equity value. Bankruptcy is only triggered when the firm cannot meet the required coupon payments by issuing additional equity, which is essentially when the value of equity reaches zero. It is possible to determine when this point is reached by seeing it as an optimal stopping time problem which can be solved using the smooth-pasting condition. Essentially this trigger consider that the equity holders have an option to abandon the firm, which further justifies the usage of a first passage time model. Alternatively, in the protected debt case the safety covenant dictates that at any time the firm value must be above the face value of debt or else the firm defaults.

While the model presented in Black and Cox (1976) was an improvement over the initial contingent claim models of Black and Scholes (1973) and Merton (1974) due to the more realistic default conditions, it didn't completely abolish all of their limitations. Specifically, it assumes that the interest rates are constant and that the strict absolute priority holds. The strict absolute priority implies that equity holders do not receive anything in case of default and that senior bondholders will only lose capital when equity holders and any junior debt holders receive zero. The goal of Longstaff and Schwartz (1995a) was to improve upon the model presented in Black and Cox (1976) by including interest rate risk. Additionally, the model also allows for deviations from the strict absolute priority. The paper presents simple closed-form solutions for valuing both risky fixed-rate and floating-rate debt. Starting with the risky fixed-rate debt, the model uses a single variable which provides a summary measure of default risk in the firm (X). With X being the ratio between the total value of the assets of the firm (V) and the barrier level (K). The importance of this is that the risky debt can be valued without separately specifying V and K, which simplifies the practical implementation of the model. Because X is a sufficient default risk statistic in the model, it is no longer necessary to condition on the pattern of cash payments to be made prior to the maturity date of the bond. Simply put, the assumption is that as soon as there is some form of financial distress the firm will default on all of its debt. Due to this, it is possible to value bonds by conditioning them directly on X which leads to the implication that coupon bonds can be valued as simple portfolios of discount bonds. Another important implication of the results in the risky fixed-rate debt section of the paper, is that credit spreads can vary significantly between firms with similar default risk when the firm's assets have different correlations with changes in interest rates. This could explain why bonds with similar credit ratings but in different industries or sectors have a very different credit spread. The results found in the risky floating-rate debt section of the paper show that this type of debt is fundamentally different from the fixed-rate debt. In contrast with fixed-rate debt, the price can be an increasing function of the bond's maturity. This is due to the mean-reversion property of the short-term risk-free rate as used in the payoff function. Additionally, the value of the floating-rate coupon payment can be an increasing function of the risk-free rate. These features of the floating-rate debt demonstrate that the correlation between interest rates and the returns of a firm can play a significant role in valuing risky corporate debt.

Thus far each of the models described have an assumption in common, namely that the assets of the firm represent a traded security. This is significantly different in the model introduced by Goldstein et al. (2001). Here it is instead assumed that there is only a traded contingent claim on the firm, in other words: equity. The paper further proposes an optimal capital structure model that deviates from the static capital structure decision. Here firms are able to adjust, mainly upwards, the amount of debt issued. The paper notes two immediate consequences. First, firms will choose to have a smaller amount of debt initially which may explain why most static models predict a higher level of debt. Second, the risk associated with the debt, at any level of debt, is higher as the bankruptcy threshold rises with the level of outstanding debt. Which in turn could explain why static models predict yield spreads that are too low. A fundamental change in the model is that the tax benefits of leverage are no longer modeled as an inflow of funds but rather as a reduction of the outflow of funds. This change solves the problem where an increase in taxes is causing an increase in equity value as seen in Leland (1994). In contrast, the model now predicts equity prices to be a decreasing function of the tax rate. Another important difference between the traditional framework and the one suggested in Goldstein et al. (2001) has to do with the risk-neutral drift. The paper explains that the traditional framework seems to significantly overestimate this risk-neutral drift, which leads to a downward bias in the probability of bankruptcy. The models that use this traditional method have to compensate for this by implementing unrealistically large bankruptcy costs in order to obtain yield spreads more in line with empirical findings. This is resolved in the framework proposed by Goldstein, which implies a lower risk-neutral drift that predicts a higher probability of default and thus leads to a lower optimal leverage ratio.

The work done for this thesis is primarily based on a model developed by Silva (2017). In this dissertation, Silva describes a cash flow from operating activities (CFO) based model that can ultimately be used to derive the price of any financial asset as a contingent claim on the project; including equity, debt, CDS and European-style options. It takes into account the shortcomings of the previously mentioned models, such as the value of the project being a non-tradable asset, and makes some additions. First of all, the model replaces the commonly used earnings before interest and taxes (EBIT) with the CFO as its state variable and it introduces a fixed costs component. This fixed cost component can, for example, be in the form of capital expenditure (capex), the required outflow of funds in order to maintain a steady project growth. Silva also adds the possibility of a sudden negative jump of fixed size in the firm's ability to generate earnings.

A closely related field to the contingent claim models is the so-called real options theory. The work by Black and Scholes (1973) and Merton (1974) opened the door for others, such as Myers, to further incorporate the option pricing theory into firm valuation. Myers (1977) argued that most firms are valued as a going concern and that this value is a reflection of the expectation that the firm will continue to invest in itself for the foreseeable future. This expectation does not dictate the amount the firm would invest, if at all, as this depends on the firm's future performance. Myers thus concludes that a portion of a firm's value must come from the option the firm has to make future investments on possibly favorable terms. Myers extends this theory by coining a new term: "real-options". Real-options are opportunities to purchase real assets on possibly favorable terms. These are mostly directly related to managerial flexibility, a portion of firm value which a standard discounted cash flow (DCF) analysis fails to incorporate. Van Aarle's paper shows that DCF analysis and real-options can be used in a complimentary way to improve the overall firm valuation quality. By using real-options the analysis can take into account the value of flexibility in an operational sense as well as strategic opportunities (van Aarle, 2013). This further establishes the interpretation that equity holders essentially hold an option to sell their project for the amount of debt owed and that they are dynamically assessing whether or not to exercise this option.

2 | The model

2.1 The asset process

Free cash flow to equity (FCFE) is a commonly used equity valuation model. FCFE measures the amount of cash available for the equity holders after the firm pays for all expenses, net debt and investment costs. The remainder can then be paid out to the equity holders in the form of a dividend payment or a stock repurchase. Alternatively, the firm can choose against distributing the cash and instead hold on to it increasing the size of their cash accounts. This ability to choose how to distribute the excess cash to equity holders is one of the main differences with the dividend discount model. This difference between the models is important, especially recently because empirical evidence suggests that many firms currently prefer stock repurchases over dividends (Grullon and Michaely, 2002) (von Eije and Megginson, 2008). Three potential explanations for this shift are described. The first being taxes; capital gains are typically taxed at a lower rate than dividends. The second has to do with the potential for firms to take advantage of a misvaluation by the market and the last is that the firm gives the equity holder a choice; receive cash or hold your shares. A choice which is made for the equity holder in the event of dividends being paid. The FCFE can be used in the equity valuation process to determine the firm value. This can be done through the assumption that the firm's value equals the present value of all future FCFE. A negative value for FCFE is possible, it implies that the firm needs to raise (or has raised) additional equity for the period. While FCFE is not a standard reporting figure, unlike dividends, it can be computed using a firm's financial statements. A commonly used method to compute the FCFE is the following:

$$FCFE_t = CFO_t + CFI_t + d_t, \tag{2.1}$$

where CFO_t is equal to the cash flow from operating activities, CFI_t represents the cash flow from investing activities and d_t stands for the net debt increase. Whereas CFO tends to be positive, CFI is typically negative. CFI is a measure of how much the firm spent on acquiring capital assets such as property, plant and equipment (PP&E) or other long-term investments in other firms. When the opposite happens, PP&E and/or long-term investments are sold, CFI turns positive. In this equity valuation model, FCFE is considered to be a growing perpetuity, or infinite horizon discrete time deterministic trend process. As with the Gordon Growth Model, the discount rate must be larger than the free cash flow growth rate to avoid an infinite value for equity. For this thesis it is important that the model properly incorporates risk, meaning that the FCFE model is considered to be a continuous time stochastic process with a finite horizon. Before elaborating on how this changes the FCFE model, let us add and subtract to equation (2.1) the total fixed costs consisting of selling, general and administrative expenses, FC, and interest expense, hereafter referred to as the coupon rate of the firm, c, times the firms total liabilities, L_t:

$$FCFE_t = \Big(CFO_t + FC_t + cL_t\Big) + \Big(CFI_t - FC_t - cL_t\Big) + d_t \tag{2.2}$$

To simplify the equation, we can denote the first term in brackets as δ_t (our state variable). By adding and subtracting the fixed costs and total liabilities to the CFO and CFI, respectively, we ensure that a positive value for δ_t is obtained. Next, we split the second set of brackets into two separate components. CFI_t plus FC will be referred to as q, a constant, and the interest rate expense will be written as cL, as L is also assumed to be constant. We will also make the assumption that throughout the life of the firm no new debt is issued, this means that the final term, d_t, is dropped as it equals zero. This assumption implies that the firm's expected leverage ratio will decreases over time, meaning that if the firm does not default at the beginning it probably never will. Important to note is that interest expense is treated as an operating expense following U.S. GAAP reporting standards. This results in the following equation:

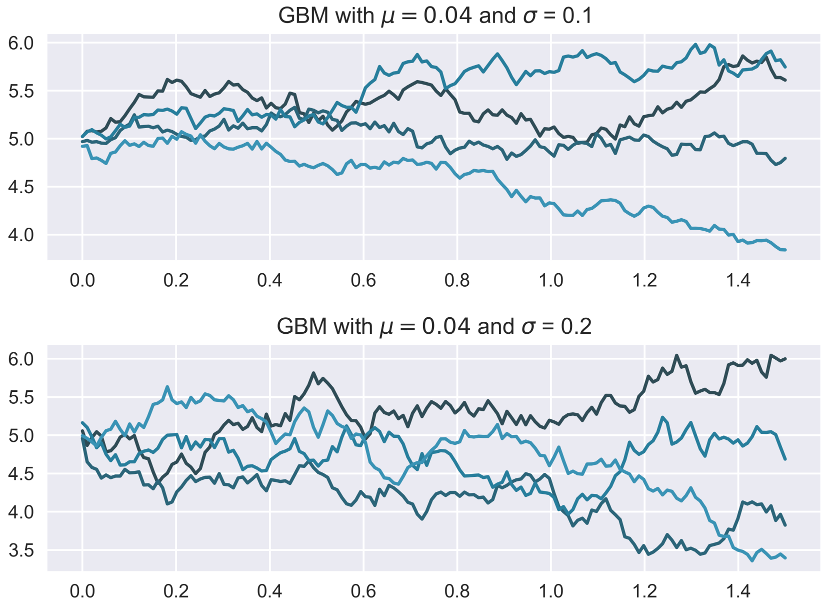
$$FCFE_t = \delta_t - q - cL. \tag{2.3}$$

Assume that the dynamics for δ_t are given by the following differential equation:

$$\frac{d\delta_t}{\delta_t} = \mu_\delta dt + \sigma dW_t^{\mathbb{P}} \tag{2.4}$$

Where $\{W_t^{\mathbb{P}}, t \geq 0\}$ is a standard Wiener process under the physical measure $\mathbb{P}$, μ_δ is the instantaneous growth rate of the firm's cash flows, σ is the instantaneous volatility of the cash flows, and α represents the drift term; all of which are exogenously determined constants. Furthermore, the model assumes that q is fixed. Equation (2.4) is a stochastic differential equation known as a geometric Brownian motion (GBM), used by Black and Scholes (1973) to model stock prices. In words it means that at each point in time the continuous compounding growth rate of the state variable δ_t, follows a normal distribution with mean $\mu_\delta \Delta t$ and variance $\sigma^2 \Delta t$. The GBM is special in that it can only obtain positive values and holds the Markov property, meaning that future values can be based solely off of the present state and that past values are irrelevant. These are properties which make the GBM appropriate for modeling stock prices. However, the downside is that GBM

Figure 2.1: An illustration of the geometric Brownian Motion for $\mu = 0.04$ with $\sigma = 0.1$ and $\sigma = 0.2$.



assumes a constant volatility and no discontinuity, the opposite of what is empirically observed. Figure 2.1 presents an illustration of the GBM for two different values of σ with $\mu = 0.04$, showing how volatility impacts the process. The figure shows that the interval of values is wider but there are no jumps, which would show as discontinuities.

As mentioned before, the FCFE model that is commonly used, discounts all of the cash flows until infinity. The model used in this thesis is different as it allows for $FCFE_t$ to change. This introduces a new dynamic where the possibility exists that δ_t becomes less than $q - cL$ and thus negative. A negative δ_t means that the firm has to raise more capital and to do so in this model, the firm has to turn to its equity holders. As long as they inject the required capital, the firm continues as normal. However, if the value of equity after injecting capital is lower than the capital injection itself, the equity holders will choose to abandon the firm which means the firm defaults and incurs distress costs such as legal fees and distressed asset sales. The model defines time τ as the first point in time where δ_t passes through $\overline{v}$, the default barrier. As will be explained later in this thesis, in this model equity is valued as the discounted sum of future FCFE until time τ and not up to infinity which contrasts with the standard FCFE model. As q and cL are constants, so is $\overline{v}$. This default barrier is determined endogenously based on the smooth-pasting condition which is further explained in section 2.2. In case of default, the firm receives βA_τ, where A_τ is equal to the discounted present value of all δ values in perpetuity. This can be written as follows:

$$A_\tau = \frac{\overline{v}}{r + \overline{m}\sigma - \mu_\delta}, \tag{2.5}$$

which implies that $\mu_A = r + \overline{m}\sigma$ meaning that the expected return for the project is comprised of the risk-free rate and a premium of $\overline{m}$ per unit of volatility risk. As r and $\overline{m}$ are constants, μ_A is also assumed to be constant. In order to better understand this, imagine the following. Project Y with value A_t generates δ_t for an infinite amount of time. Project Y is held by firm X for as long as the firm survives. As soon as δ_t falls below the threshold $\overline{v}$, the firm is closed and the project sold to a competitor and the cycle repeats. The β represents that the stakeholders of the firm only receive a share of the project value after it is sold upon default occurring. With the assumption that the regular pecking order holds, shareholders will only receive something if βA_τ is higher than L. In this thesis we will assume, for simplicity, that β will always be sufficiently low so that the shareholders receive nothing in case of the firm defaulting.

2.2 Equity

The previous section introduced the stochastic asset process and detailed how the asset value is determined as a function of the project's ability to generate cash flows. The next step is to calculate the price of the firm's equity. The value of equity in this model is made up of three components: 1) the firm's cash holdings at time t_0 (i.e. $Cash_0$), taxed the moment it is distributed; 2) the after-tax present value of all future dividends up to the moment of the firm halting its activity (i.e. $(1 - t^{-Div})Div_0$) [1]; and 3) the remainder after selling the project at time τ having paid all of the firm's liabilities after taking into account all external claimants to the project (i.e. $EqRecov_0$), however, as mentioned in the closing of the previous section, in this thesis we are assuming that $EqRecov_0 = 0$. This can be written mathematically as follows:

$$E_0 = \left(1 - t^{-Div}\right)\left(Cash_0 + Div_0\right), \tag{2.6}$$

The before-tax value of dividends can be said to equal the discounted sum of all future cash flows for as long as the firm exists, minus the discounted sum of all future coupon and fixed cost payments for as long as the firm exists. This can be represented by the following equation:

$$Div_0 = Payout_0 - Coupon_0 - FixedCosts_0. \tag{2.7}$$

The terms $Payout_0$, $Coupon_0$ and $FixedCosts_0$ represent equations 3.10, 3.15 and 3.16 as derived in Silva (2017) and can be found in further detail in Appendix A. The addition of $Cash_0$ in the equity formula is worth paying additional attention to. The intuition behind

[1] Because CFO is used as the state variable, there is no need to take into account corporate taxes again.

this is that in the past years cash holdings in firms have significantly increased. Bates et al. (2009) shows that between 1980 and 2006 the cash-to-assets ratio of US firms has more than doubled. There are many explanations for this change, of which one is that nowadays many firms hold cash in foreign subsidiaries. As these reside in countries with favorable tax regimes, these firms face a high tax costs for repatriating their cash holdings abroad. Regardless of the reason, it is especially now important to take into account the cash flow at time t_0 in a DCF model in order to fairly value companies holding high amounts of cash. Whereas $Cash_0$ is a value obtained from the data collected for each of the firms, our value for Div_0 is calculated. As described at the end of section 2.1, the project owned by the firm continuously generates δ_t. To keep the project running there is a required continuous investment of fixed costs, q, and obligatory payment of coupons to debtholders. During the lifetime of the project, for as long as δ_t is sufficient to cover the coupons and fixed costs, shareholders receive the difference. In case δ_t is insufficient the shareholders inject capital into the firm to avoid bankruptcy. Assuming that the model has no information issues and that the shareholders have no liquidity constraints, it is reasonable to say that the shareholders choose the time of default τ strategically. This predicament can be viewed as an optimal stopping time problem and can be expressed mathematically as:

$$\sup_{\tau \in \tau[0,+\infty]} E_0(\tau) \tag{2.8}$$

where $E_0(\tau)$ is given by equation (2.6) as a function of τ. This optimal stopping time problem can be solved using the smooth pasting condition. As such, $\overline{v}$ is chosen to satisfy the following:

$$\left.\frac{\partial E}{\partial A}\right|_{A=\overline{v}} = 0. \tag{2.9}$$

In practice this means that $\overline{v}$ is calculated by first taking the derivative of equation (2.6), substituting A_t by $\overline{v}$ and then equating to 0. Doing this gives derivatives of each of the terms making up equation 2.7, which in turn are used to calculate $\overline{v}$, see the equation in Appendix A.2. $\overline{v}$ is a function of $\overline{m}$ and the two show an inverse relationship; the higher the price of risk, the lower the default barrier. As the indicator of the price of risk, represented by $\overline{m}$, increases the value of $\overline{v}$ alongside the firm's asset value decreases and the two become closer. The intuition behind this is that when the market price of risk increases, the shareholder is less willing to inject capital in the firm. The next chapter presents a detailed explanation of how the dataset was prepared and explains how to calculate several of the inputs mentioned in this chapter.

3 | Model inputs

This chapter presents the data used for this thesis, detailing how it was obtained, cleaned and otherwise prepared to run the model. In addition, extra attention is paid to the calculation of several important inputs as well as how the model is executed in practice.

3.1 Dataset

This thesis focuses on the US market and uses all non-financial S&P500 firms as the base dataset. After data collection, the dataset contains 406 firms in total. All of the necessary accounting and market data is collected at annual frequency from Thomson Reuters for the period between 1998 and 2017, the equity data obtained refers to the end of the year. Cleaning the data starts by changing $\#N/A$ into zeroes. The cash flow from investing activities (CFI) is then smoothed by using a moving average, the logs of the moving averages are computed after which the fitted values are calculated. Next, the state variable δ_t, fixed costs q and interest expense cL are calculated as described in chapter 2. A short example of the data can be found in table 3.1 below. The firms that show a negative value for δ_t at any point in the time series, or those with missing values, are dropped as the model does not work in case of a negative δ_t or missing value. This leaves the dataset with 326 firms for further filtering.

Table 3.1: An example of data found in the sample.

Fiscal Year	Ticker	δ_t (\$)	q (\$)	L (\$)	c_t (%)	Equity (\$)
1998	APD.N	1,796.6	1,568.59	4,813.8	3.39	9,172.18
1999	APD.N	1,938.6	1,623.13	5,146.6	3.09	7,696.04
2000	APD.N	2,085.4	1,713.59	5,333.7	3.69	9,401.51
2001	APD.N	2,009.4	1,757.43	4,860.3	4.66	10,657.97

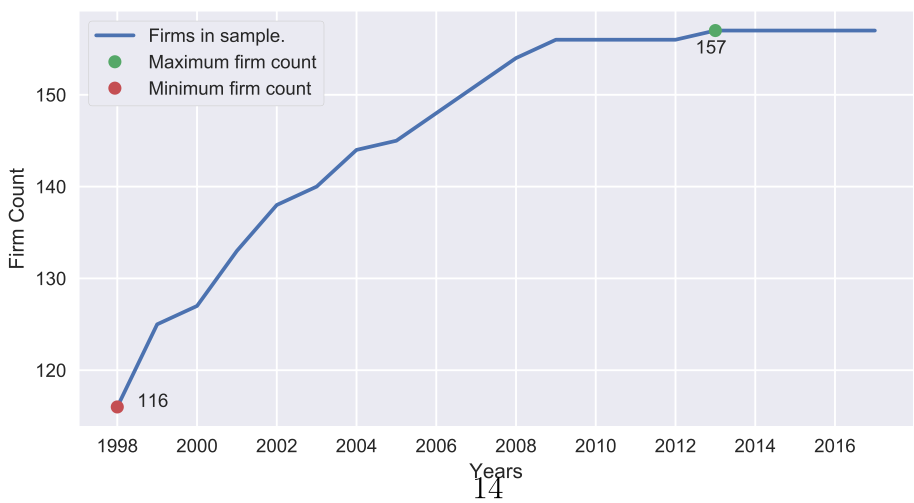
Having cleaned the dataset it is now important to restrict the firms in our sample to those that meet the following 3 criteria. First of all, we test the correlation between the firms' state variable and equity, this filtered out a total of 18 firms showing a negative correlation. The reason for this is that the model assumes that as the state variable increases the equity increases. When the correlation shows that the opposite is true, then the main assumption behind the model is false. Keeping those firms in the dataset would be superfluous. Next, we test the remaining firms in the sample for mean reversion after which the sample is restricted to firms that do not show mean reversion in the state variable. This test causes 39 firms to be dropped from the sample. The rationale being that the model uses the geometric Brownian Motion which is a stochastic process, as seen in figure 2.1, that is not mean reverting. In order to filter these firms, we perform the following regression using robust least squares:

$$\frac{\Delta x}{x} = \beta_0 + \beta_1 \frac{1}{x} + \tilde{\varepsilon}, \tag{3.1}$$

where x represents our state variable, β_0 and β_1 are the coefficients of the regression equation to be estimated and $\tilde{\varepsilon}$ is a zero-mean error term. When β_0 is negative and statistically significant at $\alpha = 5\%$, then the firm shows mean reversion. This method is the same as the one used by Longstaff and Schwartz (1995b) and Sarkar and Zapatero (2003). The last way to filter the data is by performing a normality test. The objective being to filter out firms whose returns do not appear to come from a normal distribution so that the sample does not contain extreme outliers. This is done by using the Shapiro-Wilk test, which tests the null-hypothesis that the sample came from a normally distributed population. The test produces a p-value which, if less than $\alpha = 5\%$, leads to the null-hypothesis being rejected and the firms removed from the sample. If it is larger than $\alpha = 5\%$, then the null-hypothesis cannot be rejected. A total of 112 firms are filtered from the sample using this test.

After cleaning and filtering the dataset using all three tests, we end up with 157 firms in our dataset. The number of firms per year is not exactly the same as we start with 116 firms. This variation can be explained due to the fact that the sample data was selected based on the S&P500 index in its current state. Some firms were added to the index throughout the period while others were removed for various reasons such as mergers or simply ceasing to exist. This does introduces some survivorship bias. Figure 3.1 visualizes the change of the number of firms in the sample per year, indicating the minimum and maximum number of firms.

Figure 3.1: Total number of firms in the sample per year.

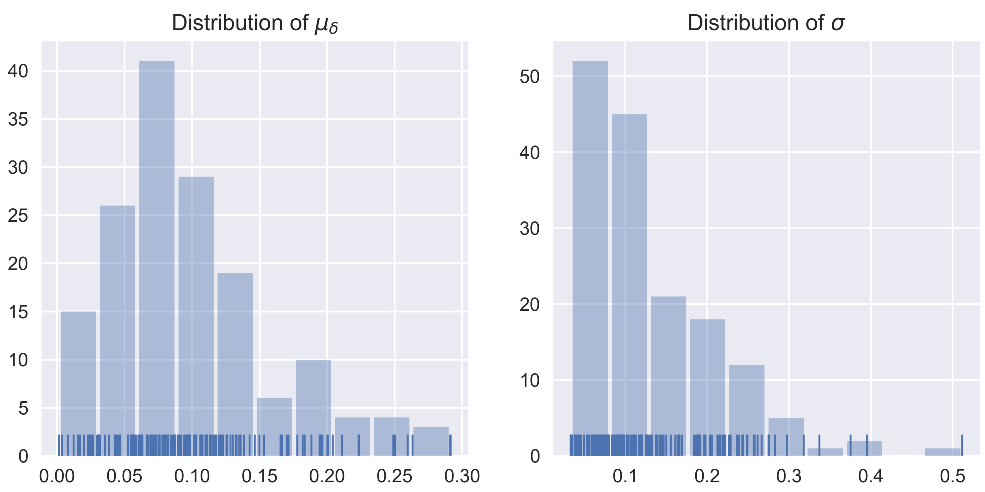


3.2 Inputs

Most of the inputs for the model are easily obtained through simple calculations with accounting data for each firm. For the risk-free rate r, we use the average yield on the 10-year Treasury bond for the period between 1998 and 2017, in accordance to our sample's time series. We assume a constant dividend tax rate of 20%. Two important parameters are μ_δ and σ which we calculate using the same method as described in Brigo et al. (2007). In Brigo's paper the difference of the log of the state variable δ_t are regressed on a constant. This thesis opts to use a robust linear regression instead of a regular linear regression. The rationale being that with a relatively small sample size for each firm potential outliers might have a noticeable effect on the results and a robust linear regression can reduce the impact of those outliers. The standard error of the residuals then gives the value for σ which in turn can be used to compute μ_δ as follows:

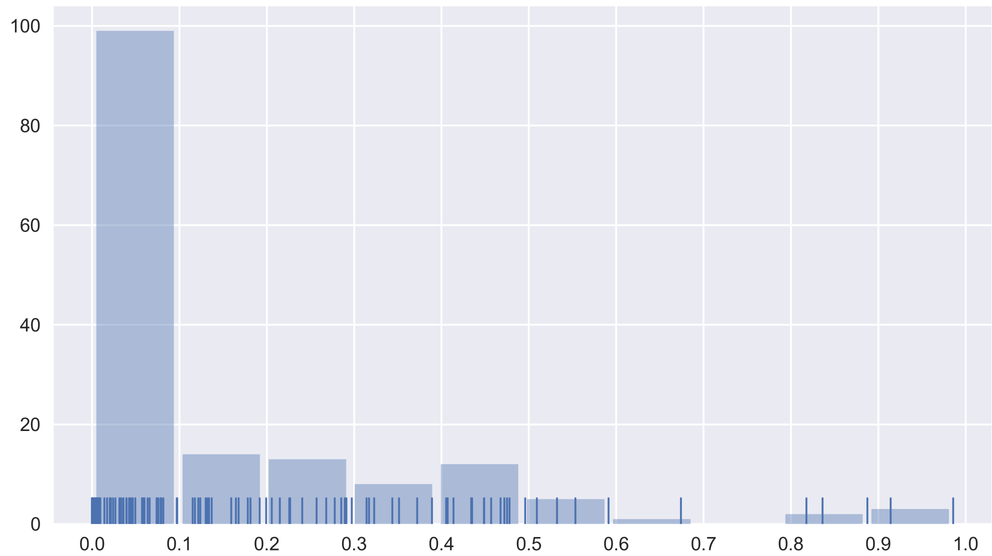
$$\mu_\delta = \beta_1 + \left(0.5 * \sigma^2\right) \tag{3.2}$$

Figure 3.2: Histogram showing the distribution of μ_δ and σ as computed by the robust linear regression.



Acceptable values for σ were obtained using this method as can be seen in figure 3.2. However, that same figure shows the regression estimated very high values for μ_δ, which reflects the growth rate observed during these 20 years, but may be a bad estimator of the expected future growth rate. In addition, the p-values associated with μ_δ were very high and, using $\alpha = 5\%$, statistically insignificant for 71 firms out of 157 firms as is visualized by figure 3.3. From the numerical analysis of Silva (2017) we can conclude that changing the μ_δ up or down by $1p.p.$ has a large effect on the value of both the firm's assets and equity. This means, however, that using values that are likely overestimated would be detrimental to the overall result of the model. In order to solve this problem, we further simplify the model by assuming that all firms have a constant μ_δ of 4%. This growth rate estimation is based on long term growth and inflation expectations and is commonly used in the industry as a growth rate proxy. In reality it is very challenging to accurately estimate the expected future growth rate of a firm. The effect of μ_δ on $\overline{m}$ will be analyzed in more detail in section 4.3.1.

Figure 3.3: Histogram showing the distribution of p-values corresponding to μ_δ as computed by the robust linear regression.



3.3 Model execution

Having collected all the data and calculated the necessary inputs, it is now time to run the model. As the objective of this thesis is to compute an indicator of the market price of risk; the final result we are looking for is $\overline{m}$. This represents an indicator of how volatility risk is priced in the market. The higher $\overline{m}$, the higher the compensation demanded by the market for holding the risky asset instead of the risk-free asset. In contrast to what is assumed in the model, the value of $\overline{m}$ is not considered to be constant. Instead, the value of $\overline{m}$ is estimated at each moment in time assuming that our model is correct and market agents estimated the values of μ_δ and σ that we have estimated. This is done by solving the following equation:

$$Equity = E_0 - EquityObserved, \tag{3.3}$$

where *EquityObserved* is equal to the market capitalization of the firm and E_0 represents equation 2.6, which is a function of $\overline{m}$.

This thesis uses the *newton* function of the Python package *scipy.optimize* which is an implementation of the secant root finding method. The algorithm tests values until it finds the root of the function. This algorithm requires an initial value, x_0, for $\overline{m}$ to get started. It is important to remember that μ_A has to be higher than μ_δ to avoid infinite asset values. As the equity function is monotonic on $\overline{m}$ the absolute distance to the root is not important. What this means is that as $\overline{m}$ increases, equity will always decrease. Regardless, in order to successfully run the model x_0 has to be computed by subtracting r from μ_δ, dividing by sigma and then incrementing the value by 0.001. The complete Python implementation of the model and its execution can be found in Appendix B. The next chapter presents the results and analysis.

4 | Results & analysis

Chapter 2 explained in detail how the model works in theory by presenting and explaining the necessary equations. Chapter 3 presented the inputs required for the execution of the model, how to compute these inputs as well as explaining how to run the model in practice. The following chapter showcases how we processed the results obtained after running the model. The results are analyzed as is and also broken down across sectors. We then highlight the effect of some of the model's assumptions by showing the sensitivity to changes in these assumptions and lastly, we analyze the coefficient of variation across time and also by sector.

4.1 Processing the results

The model returns the risk pricing indicator per firm, per year. In order to present the $\overline{m}$ values for the U.S. market we have to process this data further. First of all, the sample contains firms with different sizes in terms of market capitalization. We take this into account by calculating a weighted average using market capitalization.

Moving on, we can see that there is a significant increase in total firms throughout the sample, as visualized by figure 3.1. Now imagine that $\overline{m}_1$ is 0.5 and $\overline{m}_2$ is 0.6, where $\overline{m}_1$ and $\overline{m}_2$ are the weighted average risk price indicators. Next, at time 1 there are 100 firms and at time 2 the sample contains 110 firms. As such, it would not be correct to simply state that $\overline{m}$ increased by 20% because the sample contains different firms. To take this growing number of firms into account, as well as firms potentially leaving the sample, we implement an indexation method. The first year, t, is the base value, which is obtained by simply calculating the weighted average $\overline{m}$. For the second year, $t+1$, we first select the firms in common between period t and $t+1$ and compute the $\overline{m}$ growth, g, using the following equation making sure to use only those firms in common between the two periods:

$$g = (\overline{m}_2 - \overline{m}_1)/\overline{m}_1, \tag{4.1}$$

Then we calculate the indexed $\overline{m}_2$, which we can call $\overline{m}_2^*$, as follows:

$$\overline{m}_2^* = \overline{m}_1(1+g). \tag{4.2}$$

This computation is repeated until an $\overline{m}$ value is obtained for every year in the sample. While indexation controls for entrances and exits in the calculation of $\overline{m}$, it does not eliminate the survivorship bias. The $\overline{m}$ is calculated only based on firms present in the current composition of the index and does not include all the firms present before.

4.2 Risk pricing indicator

4.2.1 Expectations

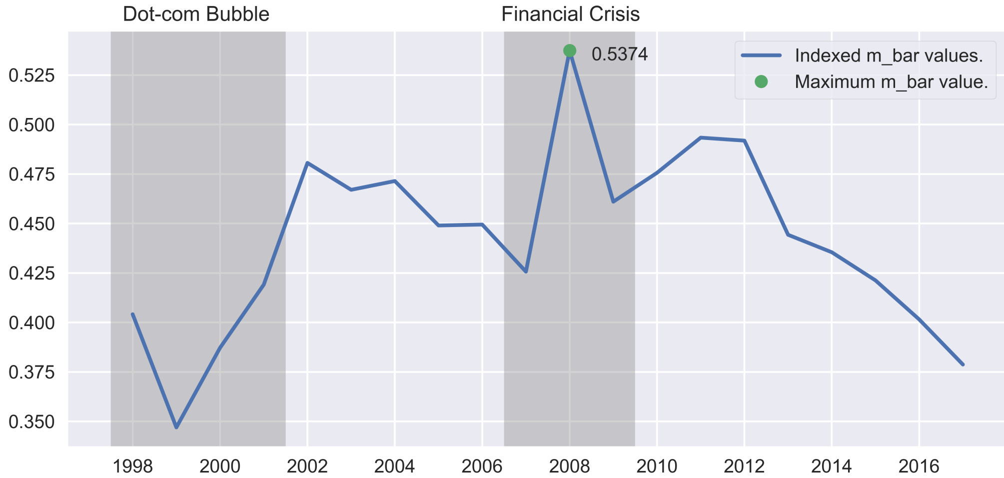
Before showing the results, it might be interesting to look at what we could expect from the model. There are two major economic events that took place between 1998 and 2017, which we can expect to have had an impact on the price of risk in the U.S. market. Their effect should thus be visible in the results of the model.

The first of two major economic events is the well-known dot-com bubble which lasted roughly from 1997 until 2001. In this period many internet-based firms were started as a result of extreme growth in internet usage. The value of the NASDAQ Composite Index, which included many of these firms, skyrocketed and peaked in value on March 10, 2000. After the burst of the bubble many firms ceased to exist while others lost more than three quarters of their value. The second major economic event is the financial crisis in 2007 and 2008, a lot has been written about this period but the main take away is that this financial crisis has had an enormous impact on the global economy. Keeping these two major economic events in mind, we can expect the risk pricing indicator to increase around the dot-com bubble followed by another peak around the financial crisis after which we should see a declining trend. As the coefficient of variation measures the dispersion around the mean, we can expect the coefficient of variation to especially capture the effect of the financial crisis in 2007-2008. Unlike the dot-com bubble, the financial crisis was a systemic crisis, meaning that every sector was affected to some extent. The expectation is that this period results in a higher $\overline{m}$ for everyone and thus a lower coefficient of variation as there will be less dispersion around the mean.

4.2.2 General results

Figure 4.1 shows the results of the model for all firms in a graphical form. The two highlighted areas of the plot represent the described major economic events, first the period of the dot-com bubble and then the financial crisis. The graph shows a significant growth in $\overline{m}$ value for the period corresponding to the dot-com bubble. These higher $\overline{m}$ values seem to persist slightly past the bubble only to slowly decline until right before the financial crisis. At that point the graph shows a significant bump in $\overline{m}$ from 0.425 all the way up to 0.5374. After peaking in 2008, the model shows a large drop in $\overline{m}$ value, the effect of the European sovereign debt crisis is also captured with another increase after which the return to pre-crisis levels starts from 2012.

Figure 4.1: Weighted average $\overline{m}$ values using the 10-Year treasury yield, where $\mu_\delta = 4\%$ and $r = 3.64\%$.



4.2.3 Results per sector

The sample used in the thesis contains firms from 9 different sectors. These sectors most likely have individual responses to market events and therefor it would be interesting to compare the evolution of the $\overline{m}$ value across the sectors. However, one important thing that we have to take into account before analyzing the results per sector, is that we run into rather limited sample sizes. While the complete sample has between 116 and 157 firms, using table 4.1 we can observe that most sectors only contain between 9 and 31 firms. The weights for each sector are found in table 4.2 which, when observed, reveals that there is a very heavy presence of 'Technology' firms in the sample; between 25% and 52% depending on the year. This may explain why the results for other sectors are significantly different from the global results. There is one sector, 'Telecommunication Services', which stands out in the sense that it is only represented by a single firm. The firm, and thus sector, in question can easily be called an outlier as it shows the highest indicator value of all. At its peak reaching an $\overline{m}$ value of 1.04 which is nearly double the maximum $\overline{m}$ value as observed in 4.1. Moving forward, the 'Telecommunication Services' sector will be omitted from the analysis to avoid the influence of this extreme outlier on the comparability of the remaining sector results.

Figure 4.2 shows the results of the model broken down per sector. Each of the lines represents a weighted average value of $\overline{m}$ for one of the 8 sectors and, like before, the highlighted areas represent the two major economic events. At a first glance the results show that the sector $\overline{m}$ values lie between 0.18 and 0.65, this is slightly lower at its minimum and slightly higher at its maximum compared to the general results. Next it is visible that the risk pricing indicator for each sector can vary widely. Again, this could potentially be due to the sample size being relatively small for each of the sectors. Note that the $\overline{m}$ of the 'Technology' sector grows heavily during the dot-com bubble.

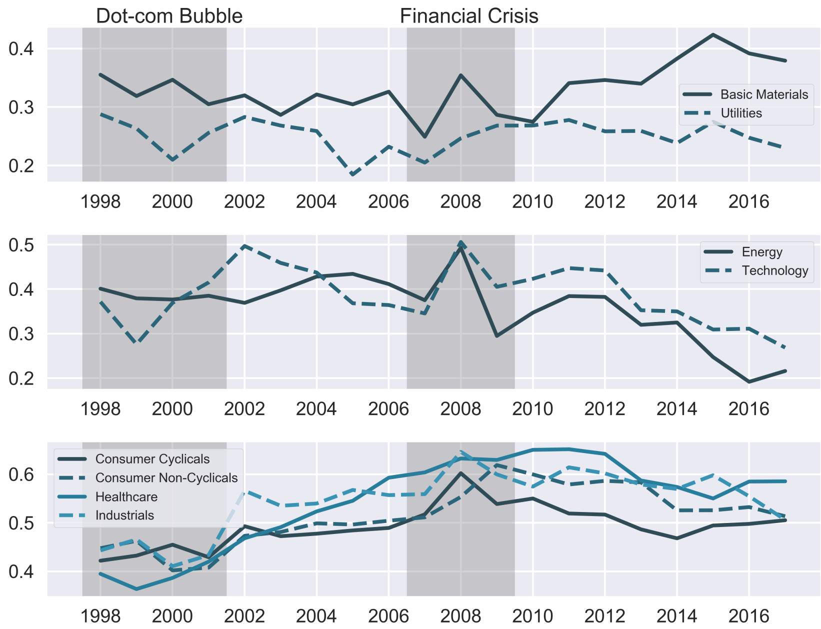
Table 4.1: Number of firms per sector, per year.

Period	Basic Materials	Consumer Cyclicals	Consumer Non-Cyclicals	Energy	Healthcare	Industrials	Technology	Telecom. Services	Utilities
1998	9	18	14	15	14	21	15	1	9
1999	9	19	14	15	16	25	16	1	10
2000	9	20	14	15	16	26	16	1	10
2001	9	21	15	15	18	27	17	1	10
2002	9	22	16	16	18	27	19	1	10
2003	9	23	16	16	18	27	20	1	10
2004	10	23	16	16	18	27	22	1	11
2005	11	23	16	16	18	27	22	1	11
2006	11	24	16	16	18	28	23	1	11
2007	11	24	16	17	18	30	23	1	11
2008	11	25	17	17	18	30	23	1	12
2009	11	25	17	17	18	31	24	1	12
2010	11	25	17	17	18	31	24	1	12
2011	11	25	17	17	18	31	24	1	12
2012	11	25	17	17	18	31	24	1	12
2013	11	26	17	17	18	31	24	1	12
2014	11	26	17	17	18	31	24	1	12
2015	11	26	17	17	18	31	24	1	12
2016	11	26	17	17	18	31	24	1	12
2017	11	26	17	17	18	31	24	1	12

Table 4.2: The weights as calculated for each sector.

Period	Basic Materials	Consumer Cyclicals	Consumer Non-Cyclicals	Energy	Healthcare	Industrials	Technology	Telecom. Services	Utilities	
1998	0.0246	0.0534	0.0800	0.1305	0.1025	0.1194	0.4075	0.0496	0.0325	1
1999	0.0244	0.0518	0.0581	0.1023	0.0742	0.1088	0.5203	0.0382	0.0217	1
2000	0.0248	0.0494	0.0764	0.1665	0.1277	0.1429	0.3156	0.0564	0.0403	1
2001	0.0258	0.0648	0.0978	0.1450	0.1271	0.1447	0.3074	0.0531	0.0344	1
2002	0.0272	0.0661	0.0929	0.1623	0.1326	0.1321	0.3032	0.0475	0.0361	1
2003	0.0320	0.0767	0.0817	0.1562	0.1296	0.1433	0.3082	0.0353	0.0370	1
2004	0.0298	0.0823	0.0754	0.1717	0.1346	0.1466	0.2822	0.0375	0.0399	1
2005	0.0290	0.0755	0.0727	0.1784	0.1306	0.1357	0.3144	0.0237	0.0402	1
2006	0.0300	0.0733	0.0721	0.2004	0.1161	0.1306	0.3095	0.0240	0.0441	1
2007	0.0378	0.0611	0.0569	0.2212	0.1012	0.1219	0.3282	0.0233	0.0484	1
2008	0.0329	0.0619	0.0988	0.2373	0.0997	0.1253	0.2587	0.0338	0.0516	1
2009	0.0341	0.0652	0.0861	0.1818	0.0971	0.1206	0.3497	0.0235	0.0419	1
2010	0.0383	0.0752	0.0906	0.2002	0.0842	0.1369	0.3126	0.0251	0.0369	1
2011	0.0315	0.0791	0.1007	0.1958	0.0880	0.1241	0.3115	0.0274	0.0419	1
2012	0.0329	0.0839	0.0978	0.1743	0.0933	0.1266	0.3214	0.0284	0.0413	1
2013	0.0262	0.0941	0.0920	0.1631	0.0808	0.1356	0.3486	0.0239	0.0356	1
2014	0.0270	0.1007	0.0941	0.1361	0.0937	0.1389	0.3348	0.0326	0.0420	1
2015	0.0232	0.1076	0.1022	0.1040	0.1001	0.1177	0.3766	0.0309	0.0378	1
2016	0.0250	0.0937	0.0929	0.1184	0.0949	0.1268	0.3765	0.0327	0.0391	1
2017	0.0254	0.0878	0.0819	0.0941	0.1034	0.1271	0.4159	0.0269	0.0375	1

Figure 4.2: Weighted average $\overline{m}$ values per sector using the 10 Year treasury yield, where $\mu_\delta = 4\%$ and $r = 3.64\%$. Results divided in three groups as per the text.



We can divide the results into three different groups containing similar sectors in terms of the obtained results. The first group of sectors comprises the 'Basic Materials' and 'Utilities' sectors. The results of the model shows that this group has the lowest risk pricing indicator of the three and especially the firms in the 'Utilities' sector react very little to these economic events. Next, the lines corresponding to the 'Energy' and 'Technology' sector form the second group, these sectors show a trend that is similar to the one for all firms together as seen in figure 4.1. The third group of sectors, 'Consumer Cyclicals', 'Consumer Non-Cyclicals', 'Healthcare' and 'Industrials', show a tendency to have a much higher price of risk with a reaction to the highlighted economic events that is much more increasing in nature. This is especially true for the 'Healthcare' sector, for which the $\overline{m}$ seems to be growing for the longest consecutive period.

One possibility is that a higher expected growth rate influences this observation, the 'Healthcare' sector is one sector in particular which showed a higher expected growth rate. The results from figure 4.2 are used to draw a comparison between $\overline{m}$ and the expected growth rate as calculated in section 3.2. However, this comparison suggests that this is likely not the only factor as in table 4.3 we can see that firms with similar expected growth rates can have a very different result for $\overline{m}$. For example, 'Consumer Non-Cyclicals' and 'Utilities' have a similar estimated μ_δ but are very far apart in terms of $\overline{m}$.

Another potential explanation for the different reactions of the sectors could stem from the nature of the products or services that they sell and whether or not they are impacted significantly by an economic downturn. Taking a closer look at the first group, these are companies that provide basic materials for construction and the production of goods

Table 4.3: Average μ_δ per sector, as calculated using the method described in section 3.2.

Sector	Average μ_δ
Basic Materials	0.0826
Consumer Cyclicals	0.0994
Consumer Non-Cyclicals	0.0648
Energy	0.1465
Healthcare	0.0912
Industrials	0.0813
Technology	0.1247
Utilities	0.0675

as well as utilities such as electricity and water. Most of these products and services are deemed necessary regardless of the current economic situation which could explain their resilience to economic events. The second group of sectors is comprised of firms in Oil & Gas exploration, refining and production as well as semi-conductors, computer and smartphone hardware, software development and IT services. These technology firms produce products that can be considered to be a luxury good. This means that in an economic boom they should perform wonderfully and in a period of downturn they should be affected. The same goes for the firms in this group that are focused on Oil & Gas. The price of oil saw itself increase nearly nine-fold from $10.53 in 1998 to $93.85 at the end of 2007, only for the price to nosedive back down to $45.59 a year later [1]. This movement in oil prices is visible in the risk price indicator figure, with a peak at roughly 0.5 in 2008 and a large dip back down to 0.3 in 2009. The third and last group of sectors has firms with a large variety of products and services on offer ranging from clothing, footwear and retailers all the way to pharmaceuticals, airlines, freight and construction, again many of these products and services can be considered as luxury goods. A lot of the firms in these sectors are dependent on the consumer's spending to stay high and face large competition at the different price levels. The consumer's spending is affected by the state of global economy and might decide to purchase lower priced goods or hold off instead. This leads to the rising trend on the graph that doesn't lower until much later compared to the other sectors.

4.3 Sensitivity analysis

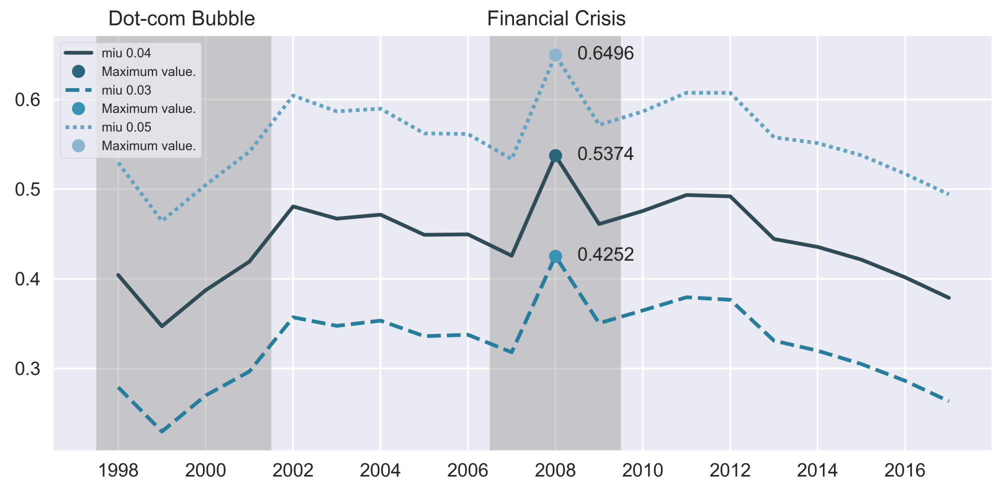
The model in this thesis is very sensitive to the risk-free rate, r, and the growth rate, μ_δ. In this section we take a look at how the model reacts to a change in these assumptions.

[1] Based on the yearly close of ICE Brent Crude futures, data sourced from the Reuters Thomson Eikon platform.

4.3.1 Growth rate

As in any discounted cash flow model, μ_δ plays a very important role and, as highlighted before, adjusting this parameter up or down by $1p.p.$ resulted in a large increase or decrease of project value in the original paper by Silva (2017). This behavior is thus also expected in the results of this thesis. In figure 4.3 we can compare the results of the model for $\mu_\delta = 3\%, 4\%$ and 5%. It becomes visible that an increase in μ_δ causes an increase in $\overline{m}$ and vice versa. While the absolute amount of change fluctuates per period, the increase is always the same as the decrease for that same period. The average value for σ is equal to 0.1277, this is very close to the average half distance between the results for $\mu_\delta = 0.03$ and 0.05. This trade-off can be explained by looking at the equation for the drift which is found in appendix A.1 as equation A.1. The growth rate is adjusted through subtraction of $\overline{m}\sigma - 0.5\sigma^2$. This means that as the growth rate increases the drift of the process also increases.

Figure 4.3: Sensitivity analysis performed by using values of 3%, 4% and 5% for the growth rate, μ_δ.

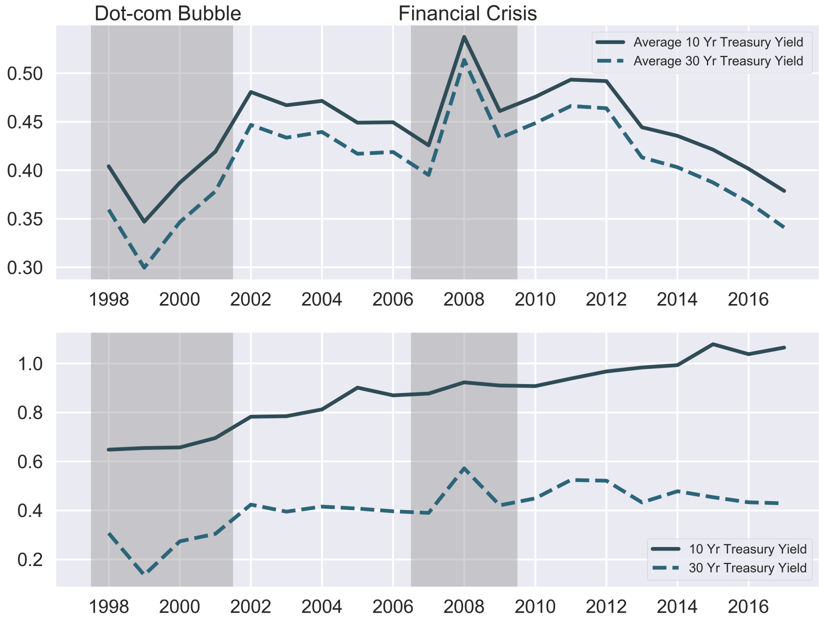


4.3.2 Risk-free interest rate

For this thesis we assume that the risk-free rate is a constant value, calculated by taking the average of the 10-year treasury yield between 1998 and 2017. Typically, the 10-year treasury yield is used as a proxy for the risk-free interest rate but the 30-year rate can also be used, as the yield for the longer period can sometimes be more stable. Figure 4.4 shows a comparison of the results using the average of the 10-year, $r_{10yr} = 3.64\%$, and 30-year treasury yield, $r_{30yr} = 4.35\%$, as well as the results using the yields corresponding to each period which we shall refer to as r^*_{10yr} and r^*_{30yr}.

When we compare the results corresponding to the average treasury yields we can see

Figure 4.4: Sensitivity analysis performed by using the average of the 10, r_{10yr}, and 30, r_{30yr}, year treasury yield in the upper plot and the 10 year, r^*_{10yr} and 30 year, r^*_{30yr} in the bottom plot.

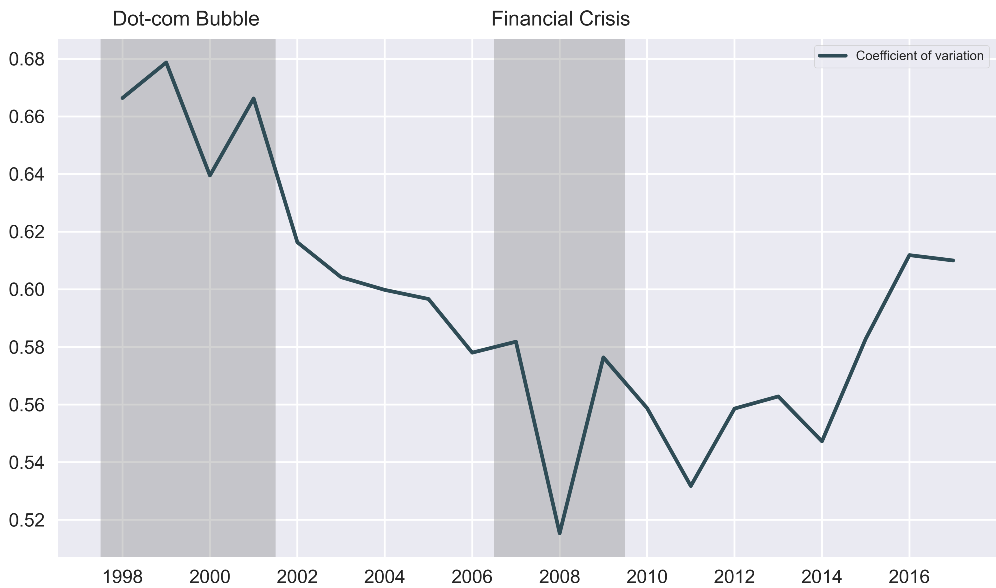


the effect a lower interest rate has on the model. By using r_{10yr}, the lower value for r, we find that the model shows an $\overline{m}$ value that seems to be shifted upwards by roughly 0.05 points in comparison to graph for r_{30yr}. However, this upward shift is not a linear one as the difference between the two lines becomes much smaller the nearer we get to the peak during crisis, then afterwards the difference gets larger again. The difference of $71b.p.$ in 2008 only causes an absolute difference in $\overline{m}$ of 0.03. This upward shift is also clearly visible for the $\overline{m}$ values when using r^*_{10yr} and r^*_{30yr}. However, we can see a more extreme movement in the graph using the r^* values, especially when looking at the dot-com bubble period the model shows an increase from $\overline{m} = 0.15$ to $\overline{m} = 0.48$. Furthermore, using r^* as our risk-free rate the model computes vales for $\overline{m}$ that do not return to lower values after the crisis. A possible explanation for the significant difference in $\overline{m}$ between using r and r^* is that in reality μ_δ and r are stochastic processes, meaning their value changes randomly over time. In this model we are simplifying this by assuming that μ_δ and r are constants. Because of the high probability that the two are positively correlated, assuming that both are constants essentially mitigates the error. When only one of the two is considered to be constant the model produces unexpected results. For example, an $\overline{m}$ that is constantly growing and ends up at a higher value in 2017 than it was at the height of the crisis in 2008.

4.4 Coefficient of variation

In this section we look at the dispersion of the market price of risk at each moment in time. To do this we use the coefficient of variation (CV), or relative standard deviation, which is a ratio of the standard deviation to the mean. The higher the CV, the greater the dispersion around the mean. The reason for choosing this metric is that it allows us to make a comparison while controlling for scale effects, this is important as the $\overline{m}$ is not constant throughout the whole period. It is computed by calculating the standard deviation of the $\overline{m}$ for each year and then dividing that by the mean of the $\overline{m}$ for that same year. Figure 4.5 shows that the CV starts its first staggered decline during the dot-com bubble period. Then as we approach the financial crisis, the dispersion around the mean reduces even further as shown by the CV falling even lower. During the 10-year period the CV falls from 0.68 to 0.515. After the crisis, the CV rises again and stagnates its growth around 0.61. The decrease observed in the CV during the systemic crisis is in line with the general idea that the correlation between sectors increases during a systemic crisis. Figure 4.5 clearly illustrates the period where there was a far more concentrated $\overline{m}$ and thus a lower value for CV. An additional note is that the results also seems to capture the effect of the European sovereign debt crisis which is visualized by the dip in CV, right after the financial crisis period ends, with its low in 2011.

Figure 4.5: Coefficient of variation (CV) per year for the sample containing all firms.



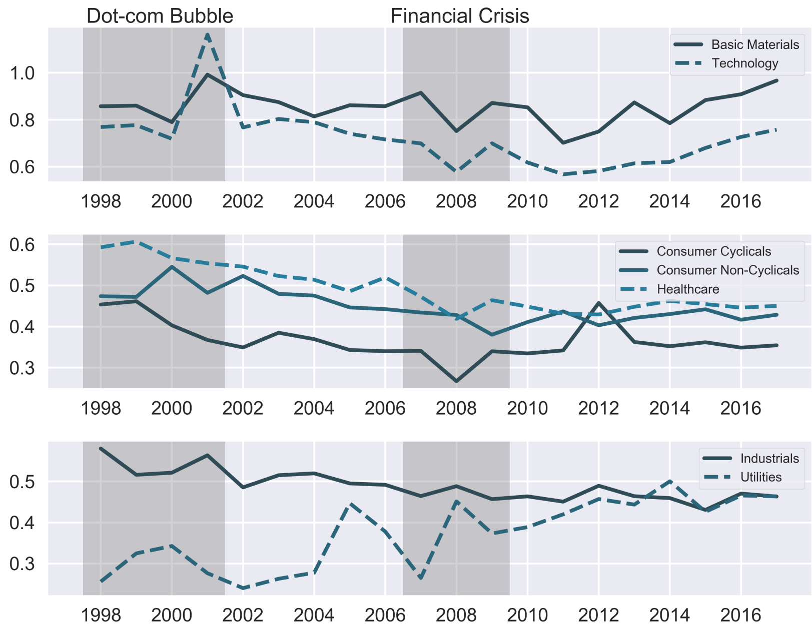
4.4.1 Dispersion per sector

When we compare the CV per sector we obtain a very different image[2]. Looking at figure 4.6, the notable outliers are the 'Technology' and 'Basic Materials' sectors which seem to have much higher CV values than the other firms. The 'Technology' sector in particular shows a big reaction to the dot-com bubble by spiking from 0.75 all the way up to 1.15. This finding seems to correspond with the fact that a lot of 'Technology' firms rapidly increased in value during this period. The 'Energy' sector starts out with a fairly high CV value of 0.73 which then falls down during the dot-com bubble. The CV value of the other sectors do not seem to show much of a reaction to the economic events and remain close together during the time period of the sample. In contrast with the results of the CV across the whole sample, we do not see the effect of the more concentrated $\overline{m}$ for each of the sectors as clearly. Instead the results suggest that some sectors are affected sooner, others later and some marginally, if at all.

The result of this thesis gives us a good idea of how different firms are affected by and how they react to major economic events. What it does not go into, but might be interesting for future studies, is to look into big differences in risk pricing for firms in the same sector. Two firms in the same sector should, in theory, be exposed to the same systematic factors and should thus also have similar prices for risk. Such a study could be used to determine potential investment opportunities together with monitoring of other early warning indicators. Combining these types of information, one could conceive a strategy specific to a particular sector by going long a firm with high market price of risk and short another that has a lower market price of risk. This results in a neutral exposure to the sector as a whole but when the two firms converge, there is a chance for profit. However, it is important to note that the difference between the firms could be because the market may have a different expectation of μ_δ for either of the two firms, unlike what is assumed in this thesis.

[2]Note that the 'Telecommunications Services' sector has, again, been omitted due to it being a single firm.

Figure 4.6: Coefficient of variation (CV) per year while splitting up firms into their respective sectors. The results have been arbitrarily divided over 3 plots for improved visibility.



5 | Conclusion

In this thesis we applied a contingent claim model to a dataset containing all non-financial firms present in the S&P500 index. The model in question is the one presented by Silva (2017), with the exception of the jump risk portion of the model for simplicity's sake. The work of Silva is an extension of Goldstein et al. (2001). It adds a fixed cost component as well as using the firm's operating cash flows as its state variable instead of EBIT. The model assumes that several parameters are constants, namely the risk-free rate, r, and the growth rate, μ_δ. The volatility, σ, is computed from the dataset. Lastly an important limitation is the assumption of constant debt, this means that the expected leverage ratio of the firm decreases over time which implies that if the firm does not default in the beginning it will probably never default. Important to note is that the model is not meant to be used to predict future values for $\overline{m}$.

Included in the time period spanning the dataset, are two major economic events. These events are expected to have an impact on the results. When looking at the $\overline{m}$ for the entire sample we see that there is a significant spike for the period of both economic events. This effect could, in part, also be seen in the per sector $\overline{m}$, however, the results suggest that not every sector is as sensitive to these economic events. The sensitivity analysis stressed the importance of μ_δ and r in the computation of $\overline{m}$. An interesting find was that the difference made by the change in r was less apparent the closer to the financial crisis of 2007-2008 we got. The final section of the analysis concerned the dispersion of the market price of risk which was computed using the coefficient of variation (CV). When looking at the results for the sample as a whole, the figure showed a decreasing trend until the financial crisis. The CV analysis captured the European sovereign debt crisis with a drop which was followed by a rising CV. It also clearly showed the effect of the systemic crisis. As $\overline{m}$ rises, the dispersion around the mean lowers implying a more concentrated $\overline{m}$. Lastly, the CV values per sector were compared which showed that the 'Basic Materials' and 'Technology' sectors had a much higher CV than the other sectors. While the 'Technology' sector showed a tremendous response to the dot-com bubble, corresponding with the sector's involvement in the economic event, the other sectors showed more stable values.

The results for the years near the end of the sample's time period showed a decreasing trend in $\overline{m}$. This indicates that investors demand less for each unit of risk, a potential sign of overconfidence in the U.S. market. In a speech by Federal Reserve Chairman Alan Greenspan in 2005, a similar sentiment is shared. Greenspan said that the vast increase in market value of asset claims at the time was in part the indirect result of investors accepting a lower compensation for risk. He warned that market participants too often view such an increase as structural and permanent, adding that their newly abundant liquidity could readily disappear. Greenspan also touched upon the risk that the reversal of a low risk premium period brings:

> "Any onset of increased investor caution elevates risk premiums and, as a consequence, lowers asset values and promotes the liquidation of the debt that supported higher asset prices. This is the reason that history has not dealt kindly with the aftermath of protracted periods of low risk premiums."
> Greenspan (2005)

More recently, the European Central Bank's (ECB) financial stability review, published in May 2018, touched upon the decline of the equity risk premium in the U.S. market. It is mentioned that the decrease can be explained by greater confidence in the future earnings growth. The influence of positive fourth-quarter earnings and the approval of lower corporate taxes in December 2017 are proposed as possible reasons for this confidence. Additionally, the report mentions that the global financial markets are displaying a higher risk tolerance and that the strong gains of U.S. asset prices is somewhat atypical. Their take is that these signs may indicate that the market perceives the that business cycle conditions will continue to improve for the foreseeable future with a low probability of a turnaround. The ECB's report shows that the possibility of significant changes in risk premiums is one of the most relevant issues for the upcoming year. European Central Bank (2018) The low $\overline{m}$ values presented in this thesis may be seen as an early warning sign for policymakers and other macroprudential authorities worldwide.

The work in the thesis is limited by some of the assumptions made, such as the constant r and μ_δ. These inputs have a significant impact on the output of the model and could potentially be over-simplifying it. A better estimation of the future growth rate of the firm could help produce better results but is a rather challenging endeavor. One also has to note the survivorship bias that is introduced by selecting firms that survived during the 20-year period as the sample is selected based on the current composition of the index. Future studies that can make use of the work done for this thesis could concern themselves with expanding the dataset, not just in length but rather also by applying the model to multiple markets in different countries. This way it will be possible to compare the responses of the different markets and possibly sectors to the economic events that occur during the period. Studying these results could give a better understanding of the differences between the global markets.

A | Equations

The model introduced in Silva (2017) has many equations that use (Greek) letters, this in order to shorten equations and allow for better readability. Some of these are used in the $Payout_0$, $Coupon_0$ and $FixedCost_0$ functions while others can be found in the Python code in Appendix B. Therefor it is important to define these auxiliary functions first after which equations 3.10, 3.15 and 3.16 from Silva (2017) can be found in appendix A.2.1, A.2.2 and A.2.3 respectively.

A.1 Auxiliary functions

A.1.1 Drift

The following equations are related to the drift of the process.

$$v^* = \mu_\delta - \overline{m}\sigma - 0.5\sigma^2, \tag{A.1}$$

which represents the log normal adjusted drift of the process.

$$\omega = v^* + 0.5\sigma^2 - r, \tag{A.2}$$

which represents the log normal adjusted drift of the process with $0.5\sigma^2$ added back and the risk-free rate subtracted.

$$a = \frac{v^*}{\sigma^2}, \tag{A.3}$$

which is the debt growth rate adjusted drift divided by sigma squared. This uses the already log normal adjusted drift.

$$R_t = \frac{\overline{v}_t}{A_0}, \tag{A.4}$$

where $\overline{v}_t = e^{(\alpha t)\overline{v}_0}$. From this follows that $R_0 = \frac{\overline{v}_0}{A_0}$. It represents the ratio of the barrier to the current project value. This value has to be < 1 or else the firm has already closed.

$$\varpi = -r, \tag{A.5}$$

which is the risk-free rate with a swapped sign.

A.1.2 Omega & Psi

The Ω and Ψ functions are defined as follows:

$$\begin{aligned}
\Omega_g^{\pm}(a,c) &= \mp\frac{\sqrt{c^2-2a}\mp c}{2\sqrt{c^2-2a}} \\
\Omega_h^{\pm}(a,c) &= \pm\frac{\sqrt{c^2-2a}\mp c}{2\sqrt{c^2-2a}} \\
\Psi_g^{\pm}(a,c) &= \mp c-\sqrt{c^2-2a} \\
\Psi_g^{\pm}(a,c) &= \mp c+\sqrt{c^2-2a}
\end{aligned} \tag{A.6}$$

A.2 Dividends

This section presents the equations used to calculate Div_0. Additionally, the derivatives of these functions are presented which are required to calculate $\overline{v}$. This is done by taking the derivative and substituting A_t for $\overline{v}$. Important to note is that (Silva, 2017) shows the mathematical proof to obtain these key functions.

A.2.1 Payout

Equation 3.10 from Silva (2017).

$$\begin{aligned}
Payout_0 = \frac{\delta_0}{\omega}\Bigg[&\Omega_h^-\left(\omega,\frac{v^*+\sigma^2}{\sigma}\right)R^{\frac{1}{\sigma}\Psi_h^-(\omega,\frac{v^*+\sigma^2}{\sigma})} \\
&+\Omega_h^-\left(\omega,-\frac{v^*+\sigma^2}{\sigma}\right)R^{2a+2+\frac{1}{\sigma}\Psi_h^-(\omega,-\frac{v^*+\sigma^2}{\sigma})}-1\Bigg].
\end{aligned} \tag{A.7}$$

Equation 3.52 from Silva (2017), the derivative of equation (A.7).

$$\begin{aligned}
\frac{\partial Payout_0}{\partial A}\bigg|_{A=\overline{v}} = \frac{\mu_A-\mu_\delta}{\omega}\Bigg\{&\Omega_h^-\left(\omega,\frac{v^*+\sigma^2}{\sigma}\right)\left[1-\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^*+\sigma^2}{\sigma}\right)\right] \\
&+\Omega_h^-\left(\omega,-\frac{v^*+\sigma^2}{\sigma}\right)\left[-2a-1-\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^*+\sigma^2}{\sigma}\right)\right]-1\Bigg\}
\end{aligned} \tag{A.8}$$

A.2.2 Coupon

Equation 3.15 from Silva (2017).

$$Coupon_0 = \frac{cL}{\varpi}\left[\Omega_h^-\left(\varpi,\frac{v^*}{\sigma}\right)R^{\frac{1}{\sigma}\Psi_h^-(\varpi,\frac{v^*}{\sigma})}+\Omega_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)R^{2a+\frac{1}{\sigma}\Psi_h^-(\varpi,-\frac{v^*}{\sigma})}-1\right]. \tag{A.9}$$

Equation 3.53 from Silva (2017), the derivative of equation (A.9).

$$\left.\frac{\partial Coupon_0}{\partial A}\right|_{A=\overline{v}} = \frac{cL}{\varpi\overline{v}}\Bigg\{-\frac{1}{\sigma}\Omega_h^-\left(\varpi,\frac{v^*}{\sigma}\right)\Psi_h^-\left(\varpi,\frac{v^*}{\sigma}\right) + \Omega_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)\left[-2a-1-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)\right]\Bigg\} \tag{A.10}$$

A.2.3 Fixed costs

Equation 3.16 from Silva (2017).

$$FixedCosts_0 = \frac{q}{\varpi}\left[\Omega_h^-\left(\varpi,\frac{v^*}{\sigma}\right)R^{\frac{1}{\sigma}\Psi_h^-(\varpi,\frac{v^*}{\sigma})} + \Omega_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)R^{2a+\frac{1}{\sigma}\Psi_h^-(\varpi,-\frac{v^*}{\sigma})} - 1\right]. \tag{A.11}$$

Equation 3.54 from Silva (2017), the derivative of equation (A.11).

$$\left.\frac{\partial FixedCosts_0}{\partial A}\right|_{A=\overline{v}} = \frac{q}{\varpi\overline{v}}\Bigg\{-\frac{1}{\sigma}\Omega_h^-\left(\varpi,\frac{v^*}{\sigma}\right)\Psi_h^-\left(\varpi,\frac{v^*}{\sigma}\right) + \Omega_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)\left[-2a-1-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^*}{\sigma}\right)\right]\Bigg\} \tag{A.12}$$

A.2.4 Endogenous barrier

The equation for computing the endogenous barrier, $\overline{v}$, is equal to the following:

$$\overline{v} = \frac{A+B}{C}, \tag{A.13}$$

where A represents equation A.10 without $\overline{v}$ on the denominator, B represents equation A.12 without $\overline{v}$ on the denominator and C represents equation A.8.

B | Python Code

This appendix contains the Python 3.6 implementation of the model. The Python code and all of the files necessary to run the model yourself can also be found on my GitHub page:

https://github.com/nobelv/msc-dissertation-credit-risk-model

B.1 Model functions

```python
# Dissertation Credit Risk Modeling
# Market price of risk implied in stocks
# Credit Risk Model functions - ignoring any jump process

import numpy as np
import scipy.stats as sp
import statsmodels.api as sm

# State variable functions - A_0, miu_A, sigma, small_r and miu_delta

def big_a_0(delta0, miua, miudelta):
    """
    The value of a security A_t at the beginning of the process is given by the function A0 = delta_0 / (miu_A - g).
    Where g = miu_delta - (lambda*jump), with jump being equal to 0, g = miu_delta.

    :param delta0: the value of the state variable at t = 0.
    :param miua: the discount rate, assumed to be constant for mathematical tractability (miu_big_a).
    :param miudelta: instantaneous growth rate of the firm.

    :return: The value of the security at time 0.
    """

    return delta0 / (miua - miudelta)


def miu_big_a(r, mbar, sigma):
    """
    The discount rate, assumed to be constant
```

```

    :param r: The risk free rate.
    :param mbar: The risk premium.
    :param sigma: The standard error of the residuals of the robust linear regression on the state variable.
    :return: The discount rate.
    """

    return float(r) + mbar * sigma


def sigma_and_miu(gvkey, statevar_dict, fixedmiu=False):
    """
    Calculates the instantaneous growth rate of the firm, the miu_delta, through a robust linear regression on the
    differences between the log of the state variable. The sigma comes from the standard error of the residuals after
    applying the robust weights.

    :param gvkey: The gvkey corresponding to the firm.
    :param statevar_dict: The dictionary containing all gvkeys and the state variable values.
    :param fixedmiu: determines whether to use a fixed value of miu_delta or not.

    :return: Returns a tuple containing miu_delta (instantaneous growth rate of the firm) and sigma (robustly weighted
    standard error of the residuals).
    """
    statevar = np.asarray(statevar_dict[gvkey])
    y = np.log(statevar[1:]) - np.log(statevar[:-1])
    x = np.ones(len(y))

    rlm_model = sm.RLM(y, x, M=sm.robust.norms.HuberT())
    rlm_results = rlm_model.fit()

    # used for debugging
    # print(rlm_results.summary(yname='y',xname=['var_%d' % i for i in range(len(rlm_results.params))]))

    sigma_calc = np.std(rlm_results.resid * rlm_results.weights)
    miudelta = rlm_results.params[0] + (0.5 * sigma_calc ** 2)
    if fixedmiu is True:
        miudelta = 0.04
    return miudelta, sigma_calc


def small_r(rate=1):
    """
```

```python
    The average of the risk free rate proxy for the time period of our sample.
    In this case the options are 10 year and 30 year US treasury yield.

    Where rate=1 specifies the 10 year rate and rate=2 specifies the 30 year rate.
    By default we use the 10 year rate

    :param rate: Parameter to specify the yield rate to be used in the model.

    :return: The yield for the specified time frame.
    """

    if rate == 1:
        r = 0.036362
    else:
        r = 0.042914
    return r


# Functions used in the equity and barrier calculation - small_omega var_pi

def omega(r, sigma, vstar):
    """
    Adding 0.5 sigma squared to the log normal adjusted drift and then subtracting the risk free interest rate

    :param r: The risk free interest rate.
    :param sigma: The standard error of the residuals of the robust linear regression on the state variable.
    :param vstar: The lognormal adjusted drift.

    :return: The log normal adjusted drift plus 0.5 sigma squared minus the risk free interest rate.
    """

    return vstar + 0.5 * sigma ** 2 - r


def var_pi(r):
    """
    In a model with jump risk this would be equal to -(r + lambda_bar), However, our model ignores jump risk.
    As such var_pi is simply the risk free interest rate with a swapped sign.

    :param r: The risk free interest rate.

    :return: The swapped sign risk free interest rate.
    """
    return -r
```

```
# Barrier and Equity functions - v_bar, payout_0, coupon_0, capex_0, effective_taxrate and div0


def v_bar(sigma, vstar, r, mbar, miudelta, couponrate, liabilities , smallomega, smalla, q):
    """
    The Default barrier. If the firm value/asset value passes is lower than this point, the
    shareholders
    give up the firm and the firm defaults.

    :param sigma: The standard error of the residuals of the robust linear regression on the state
    variable.
    :param vstar: The lognormal adjusted drift.
    :param r: The risk free interest rate.
    :param mbar: The market price of risk.
    :param miudelta: the instantaneous growth rate of the firm.
    :param couponrate: The firm's interest expense coupon rate.
    :param liabilities : The firm's total debt.
    :param smallomega: The log normal adjusted drift plus 0.5 sigma squared minus the risk free
    interest rate.
    :param smalla: Lognormal adjusted drift divided by sigma squared.
    :param q: The firm's nominal capital expenditure.

    :return: The level of the default barrier.
    """
    # Formula 3.52 - payout_0 formula where we replace A with v_bar
    a1 = smallomega
    c1 = (vstar + sigma ** 2) / sigma
    deriv_payout_0 = ((miu_big_a(r, mbar, sigma) - miudelta) / smallomega) * \
                     (big_omega_h_minus(a1, c1) * (1 - (1 / sigma) * psi_h_minus(a1, c1)) +
                      big_omega_h_minus(a1, - c1) *
                      (- 2 * smalla - 1 - (1 / sigma) * psi_h_minus(a1, - c1)) - 1)

    # Formula 3.53 - coupon_0 formula where we replace A with v_bar and isolate v_bar
    a2 = var_pi(r)
    c2 = (vstar / sigma)
    deriv_coupon_0 = ((couponrate * liabilities) / var_pi(r)) * \
                     (- (1 / sigma) * big_omega_h_minus(a2, c2) * psi_h_minus(a2, c2) +
    big_omega_h_minus(a2, - c2) *
                      (- 2 * smalla - (1 / sigma) * psi_h_minus(a2, - c2)))

    # Formula 3.54 - fixedcost_0 formula where we replace A with v_bar and isolate v_bar

    deriv_fixedcost_0 = (q / a2) * \
                    (- (1 / sigma) * big_omega_h_minus(a2, c2) *
                     psi_h_minus(a2, c2) + big_omega_h_minus(a2, - c2) *
```

```python
                    (- 2 * smalla - (1 / sigma * psi_h_minus(a2, - c2))))

    return (deriv_coupon_0 + deriv_fixedcost_0) / deriv_payout_0


def payout_0(delta0, r, vstar, sigma, bigr, smalla):
    """
    Formula 3.10.

    Discounted sum of all future cash flows as long as the firm exists.

    :param delta0: The value of our cash flow based state variable at t = 0.
    :param r: The risk free interest rate.
    :param vstar: The lognormal adjusted drift.
    :param sigma: The standard error of the residuals of the robust linear regression on the state variable.
    :param bigr: Ratio of the barrier to the current project value.
    :param smalla: Lognormal adjusted drift divided by sigma squared.

    :return: The value of the discounted sum of all future cash flows.
    """
    s_omega = omega(r, sigma, vstar)
    a = s_omega
    c = (vstar + sigma ** 2) / sigma
    aux_omg_h_min_pos = big_omega_h_minus(a, c)
    aux_omg_h_min_min = big_omega_h_minus(a, - c)
    aux_bigr_1 = bigr ** ((1 / sigma) * psi_h_minus(a, c))
    aux_bigr_2 = bigr ** ((2 * smalla) + 2 + (1 / sigma) * psi_h_minus(a, - c))

    return (delta0 / s_omega) * ((aux_omg_h_min_pos * aux_bigr_1) + (aux_omg_h_min_min * aux_bigr_2) - 1)


def coupon_0(couponrate, liabilities, varpi, vstar, sigma, bigr, smalla):
    """
    Formula 3.15

    Discounted sum of all future interest costs as long as the firm exists.

    :param couponrate: The firm's interest expense coupon rate.
    :param liabilities: The firm's total debt.
    :param varpi: The swapped sign risk free interest rate.
    :param vstar: The lognormal adjusted drift.
    :param sigma: The standard error of the residuals of the robust linear regression on the state variable.
    :param bigr: Ratio of the barrier to the current project value.
    :param smalla: Lognormal adjusted drift divided by sigma squared.
```

```

    :return: The value of the discounted sum of all future interest costs.
    """
    a = varpi
    c = (vstar / sigma)
    aux_big_omg_h_min_pos = big_omega_h_minus(a, c)
    aux_big_omg_h_min_min = big_omega_h_minus(a, - c)
    aux_bigr_1 = bigr ** ((1 / sigma) * psi_h_minus(a, c))
    aux_bigr_2 = bigr ** (2 * smalla + (1 / sigma) * psi_h_minus(a, - c))

    return ((couponrate * liabilities ) / varpi) * \
           ((aux_big_omg_h_min_pos * aux_bigr_1) + (aux_big_omg_h_min_min * aux_bigr_2) - 1)


def fixedcost_0(q, varpi, vstar, sigma, bigr, smalla):
    """
    Formula 3.16

    Discounted sum of all future fixedcosts costs as long as the firm exists.

    :param q: The firm's nominal capital expenditure.
    :param varpi: The swapped sign risk free interest rate.
    :param vstar: The lognormal adjusted drift.
    :param sigma: The standard error of the residuals of the robust linear regression on the state variable.
    :param bigr: Ratio of the barrier to the current project value.
    :param smalla: Lognormal adjusted drift divided by sigma squared.

    :return: The value of the discounted sum of all future fixedcosts costs.
    """
    a = varpi
    c = (vstar / sigma)
    aux_big_omg_h_min_pos = big_omega_h_minus(a, c)
    aux_big_omg_h_min_min = big_omega_h_minus(a, - c)
    aux_bigr_1 = bigr ** ((1 / sigma) * psi_h_minus(a, c))
    aux_bigr_2 = bigr ** (2 * smalla + (1 / sigma) * psi_h_minus(a, - c))

    return (q / varpi) * ((aux_big_omg_h_min_pos * aux_bigr_1) + (aux_big_omg_h_min_min * aux_bigr_2) - 1)


def div_taxrate():
    """
    Used to calculate the effective tax rate.

    :return: Returns the percentage effective tax rate as a float.
```

```
    """
    taxdiv = 0.20

    return 1 - taxdiv


def div0(effectivetax, payout, coupon, fixedcosts):
    """
    Calculate the value of dividends which is equal to equity assuming there is no equity recovery
     by shareholders..

    :param effectivetax: The effective tax rate for the market.
    :param payout: The value of the discounted sum of all future cash flows.
    :param coupon: The value of the discounted sum of all future interest costs.
    :param fixedcosts: The value of the discounted sum of all future fixedcosts costs.

    :return: The company's equity value according to the model.
    """
    return (1 - effectivetax) * (payout - coupon - fixedcosts)


# Drift related functions - rho, v_star, small_a and big_r

def v_star(miudelta, mbar, sigma):
    """
    Lognormal adjusted drift of the process.

    :param miudelta: The instantaneous growth rate of the project cash flows (exogeniuously
     determined).
    :param mbar: The premium per unit of volatility risk.
    :param sigma: The standard error of the residuals of the robust linear regression on the state
     variable.

    :return: The drift adjusted for lognormality.
    """

    return miudelta - (mbar * sigma) - 0.5 * (sigma ** 2)


def small_a(vstar, sigma):
    """
    Lognormal adjusted drift divided by sigma squared.

    :param vstar: The lognormal adjusted drift.
    :param sigma: The standard error of the residuals of the robust linear regression on the state
     variable.

```

```
    :return: the value of a based on v_star and sigma.
    """

    return vstar / sigma ** 2


def big_r(vbar, biga0):
    """
    Ratio of the barrier to the current project value.
    Basically has to be < 1 or the firm is already closed.

    :param vbar: The barrier value.
    :param biga0: The value of the security at time 0.

    :return: returns the distance from the barrier.
    """

    return vbar / biga0


# Auxiliary functions - omega & psi


# Simplifying function to reduce code of omega/g/h/psi funcs by placing the sqrt portion in its own
    variable
def d(a, c):
    return np.sqrt(c ** 2 - 2 * a)


# Auxiliary functions found on page 33
def big_omega_g_plus(a, c):
    return - (d(a, c) - c) / (2 * d(a, c))


def big_omega_g_minus(a, c):
    return (d(a, c) + c) / (2 * d(a, c))


def big_omega_h_plus(a, c):
    return - (d(a, c) + c) / (2 * d(a, c))


def big_omega_h_minus(a, c):
    return (d(a, c) - c) / (2 * d(a, c))


def psi_g_plus(a, c):
```

```
    return - c - (d(a, c))


def psi_g_minus(a, c):
    return c - (d(a, c))


def psi_h_plus(a, c):
    return - c + (d(a, c))


def psi_h_minus(a, c):
    return c + (d(a, c))


def g_plus(a, b, c, y):
    return np.exp(- b * psi_g_plus(a, c) * sp.norm.cdf(((- b - y * d(a, c)) / np.sqrt(y)), loc=0.0, scale=1.0))


def g_minus(a, b, c, y):
    return np.exp(+ b * psi_g_minus(a, c) * sp.norm.cdf(((+ b - y * d(a, c)) / np.sqrt(y)), loc=0.0, scale=1.0))


def h_plus(a, b, c, y):
    return np.exp(- b * psi_h_plus(a, c) * sp.norm.cdf(((- b + y * d(a, c)) / np.sqrt(y)), loc=0.0, scale=1.0))


def h_minus(a, b, c, y):
    return np.exp(+ b * psi_h_minus(a, c) * sp.norm.cdf(((+ b + y * d(a, c)) / np.sqrt(y)), loc=0.0, scale=1.0))
```

B.2 Model execution

```
# Dissertation Credit Risk Modeling
# Indicator of the market price of risk implied in stocks
# Credit Risk Model functions - ignoring any jump process

# Model Execution file, requires model_functions.py & data_functions.py!
import model_functions as mf
import statsmodels.api as sm
```

```
import scipy.optimize as sp
import scipy.stats as sps
import pandas as pd
import numpy as np
import timeit
import os

# Starting a timer
start_time = timeit.default_timer()

# Setting up dictionaries with all the data

path = os.path.dirname(os.path.abspath(""))

# Reading the data into a dataframe from our csv files.
df = pd.read_csv(path + "\Data\cashflow_data.csv", sep=",",
                 dtype={"fyear": str, "ticker": str, "firmname": str, "industry": str, "sector": str,
                        "statevariable": float, "cash": float, "fixedcosts": float, "couponrate":
    float,
                        " liabilities ": float, "equityobserved": float})

df['fyear'] = pd.to_datetime(df['fyear'], format="%m%d%Y").apply(lambda x: x.date())

df2 = pd.read_csv(path + "\Data\\rf_10yr_avg.csv", sep=';')
df2['thedate'] = pd.to_datetime(df2['thedate'], format="%Y%m%d")
df2 = df2.set_index(['thedate'])

print('Data succesfully loaded into dataframe.')

keys = df.ticker.unique()
print("Dataset contains", len(keys), "firms.")
statevar_dict = {}
cash_dict = {}
liabilities_dict = {}
coupon_dict = {}
fixedcosts_dict = {}
fyear_dict = {}
date_dict = {}
equityobs_dict = {}


# Populating our dictionaries with data specific to each company.
for key in range(len(keys)):
    k = keys[key]
    dataframe = df.loc[df['ticker'] == k]
    statevar = dataframe['statevariable']
    cash = dataframe['cash']
```

```
    liabilities = dataframe['liabilities']
    fixedcosts = dataframe['fixedcosts']
    fyear = dataframe['fyear']
    date = dataframe['fyear']
    equityobs = dataframe['equityobserved']

    statevar_dict.update({k: statevar})
    cash_dict.update({k: cash})
    liabilities_dict.update({k: liabilities})
    fixedcosts_dict.update({k: fixedcosts})
    fyear_dict.update({k: fyear})
    date_dict.update({k: date})
    equityobs_dict.update({k: equityobs})

print('Successfully populated dictionaries. Beginning model execution.')

# Executing the model

miu_delta_dict = {}
sigma_dict = {}
mbar_list = []
sigma_list = []
miudelta_list = []
miu_a_list = []
vbar_list = []

# Lists for statistical tests
sw_list = []
corr_list = []

# counters for filtering and failures
mean_rev_counter = 0
correl_counter = 0
sw_counter = 0
fail_counter = 0

# determining risk free rate type
# rf_rate = mf.small_r(rate=1)

for i in range(len(keys)):
    k = keys[i]

    # test for correlation between equity and statevar
    correlation = np.corrcoef(equityobs_dict[k], statevar_dict[k])
    correlation = correlation[1].item(0)

    if correlation > 0:
```

```

        # if correlation positive do mean reversion test
        statevar = np.asarray(statevar_dict[k])
        y = (statevar[1:] - statevar[:-1]) / statevar[:-1]
        X = 1 / statevar[:-1]
        X = sm.add_constant(X)
        rlm_model = sm.RLM(y, X, M=sm.robust.norms.LeastSquares())
        rlm_results = rlm_model.fit()

        if rlm_results.params[0] < 0 and rlm_results.pvalues[0] < 0.05:
            mean_rev_counter += 1
            # print("Firm:", k, " fails the meanrev test.", rlm_results.params[0], rlm_results.pvalues[0]) # for debug
            df = df[~df['ticker'].isin([k])]
        else:
            *_, sw = sps.shapiro(statevar_dict[k])
            if sw > 0.05:
                for n in range(len(statevar_dict[k])):
                    corr_list.append(correlation)
                    sw_list.append(sw)

                miudelta, sigma = mf.sigma_and_miu(k, statevar_dict, fixedmiu=True)
                sigma_dict.update({k: sigma})
                miu_delta_dict.update({k: miudelta})

                for t in range(len(statevar_dict[k])):
                    statevar_list = statevar_dict[k].tolist()
                    cash_list = cash_dict[k].tolist()
                    fixedcosts_list = fixedcosts_dict[k].tolist()
                    delta_0 = statevar_list[t]
                    cash_0 = cash_list[t]
                    liabilities_list = liabilities_dict[k].tolist()
                    equityobserved = equityobs_dict[k].tolist()
                    couponrate = df['couponrate'].tolist()
                    fyear = fyear_dict[k].tolist()

                    dates = date_dict[k].tolist()
                    rf_rate = df2.iloc[df2.index.get_loc(dates[t], method="nearest")].tolist()

                    # Create shorthands used in the function quadratic(m_bar)
                    sigm = sigma_dict[k]
                    r = rf_rate[0]
                    miu_delta = miu_delta_dict[k]
                    c = couponrate[t]
                    L = liabilities_list[t]
                    q = fixedcosts_list[t]
                    eq_obs = equityobserved[t]
```

```
                varpi = mf.var_pi(r)

                # Set the initial 'guess' for the x0 to be used in root finding algo
                x0 = ((miu_delta - r) / sigm) + 0.001

                def quadratic(m_bar):

                    divtax = mf.div_taxrate()
                    v_star = mf.v_star(miudelta, m_bar, sigm)
                    omg = mf.omega(r, sigm, v_star)
                    a = mf.small_a(v_star, sigm)
                    v_bar = mf.v_bar(sigm, v_star, r, m_bar, miudelta, c, L, omg, a, q)
                    miu_a = mf.miu_big_a(r, m_bar, sigm)
                    big_a_0 = mf.big_a_0(delta_0, miu_a, miudelta)
                    big_r = mf.big_r(v_bar, big_a_0)

                    payout_0 = mf.payout_0(delta_0, r, v_star, sigm, big_r, a)
                    coupon_0 = mf.coupon_0(c, L, varpi, v_star, sigm, big_r, a)
                    fixedcost_0 = mf.fixedcost_0(q, varpi, v_star, sigm, big_r, a)

                    return divtax * (cash_0 + payout_0 - coupon_0 - fixedcost_0) - eq_obs

                try:
                    mbar = sp.newton(quadratic, x0)
                    mbar_list.append(mbar)
                    miudelta_list.append(miudelta)
                    sigma_list.append(sigm)
                    miua = mf.miu_big_a(rf_rate[0], mbar, sigm)
                    miu_a_list.append(miua)

                    # calculate barrier
                    vstar = mf.v_star(miudelta, mbar, sigm)
                    omega = mf.omega(r, sigm, vstar)
                    smalla = mf.small_a(vstar, sigm)

                    vbar = mf.v_bar(sigm, vstar, r, mbar, miudelta, c, L, omega, smalla, q)
                    vbar_list.append(vbar)

                except RuntimeError:
                    print("Failed to converge after 50 iterations.", "Attempted calculation for company", k,
                          "for date", fyear[t])
                    fail_counter += 1
                    mbar = 99.99
                    mbar_list.append(mbar)
                    miudelta_list.append(miudelta)
                    sigma_list.append(sigm)
```

```
                        miua = mf.miu_big_a(rf_rate[0], mbar, sigm)
                        miu_a_list.append(miua)
                        vbar_list.append(99.99)
                        pass
            else:
                sw_counter += 1
                df = df[~df['ticker'].isin([k])]

    else:
        correl_counter += 1
        # print("Firm:", k, " fails the correlation test.", correlation) # for debug
        df = df[~df['ticker'].isin([k])]

print('Completed running the model, writing data to output file.')

keys_count = df.ticker.unique()
print("Model succesfully ran for", len(keys_count), "firms. Filtering out", len(keys) - len(keys_count), "firms.")
print(correl_counter, "firms filtered due to correlation,", mean_rev_counter, "firms filtered due to mean reversion.",
      sw_counter, "firms filtered due to SW test.")
print(" fails ", fail_counter)

mbar_list = pd.Series(mbar_list)
sigma_list = pd.Series(sigma_list)
miudelta_list = pd.Series(miudelta_list)
miu_a_list = pd.Series(miu_a_list)
sw_list = pd.Series(sw_list)
corr_list = pd.Series(corr_list)
vbar_list = pd.Series(vbar_list)

df['mbar'] = mbar_list.values
df['sigma'] = sigma_list.values
df['miu_delta'] = miudelta_list.values
df['miu_A'] = miu_a_list.values
df['Shapiro-Wilk'] = sw_list.values
df['Correlation'] = corr_list.values
df['vbar'] = vbar_list.values

df.to_csv(path + "\Model Output\model_output_10yr_avg.csv", sep=",", index=False)


# Stopping the timer
elapsed = timeit.default_timer() - start_time
print("Model executed in", '{:5.2f}'.format(elapsed), "seconds.")
```

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