

Can the trade-off theory better match empirical evidence by implementing mean reverting earnings? A theoretical study on the trade-off theory

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Abstract

Title: Can the trade-off theory better match empirical evidence by implementing mean reverting earnings? A theoretical study on the trade-off theory

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Keywords: Optimal capital structure; Trade-off theory; Leverage ratio

The traditional trade-off theory predicts a positive relation between earnings and leverage ratio, contradictory to empirical evidence which suggests a negative relationship between the two. The objective of this dissertation is to analyse whether introducing mean reverting earnings to the traditional trade-off theory turns the model results more in line with empirical evidence.

In the dissertation, I implement the Sarkar and Zapatero (2003) model, which considers mean reverting earnings before interest, taxes and depreciations in a setup similar to the proposed by Leland (1994). I use this model to calculate the optimal leverage ratio. The original input parameters presented by Sarkar and Zapatero (2003) were considered and equal results were obtained. The model was solved numerically for the optimal default barrier, optimal coupon level, and finally for the optimal leverage ratio. Furthermore, different initial earnings levels, different speeds of mean-reversion and different mean earnings-levels were considered to observe how the model handles different input values.

As argued by Sarkar and Zapatero (2003), the consideration of mean-reversion leads to a negative relationship between the optimal leverage ratio and earnings, which is in line with empirical evidence. This result should change the generalized idea that the trade-off theory is not consistent with data regarding the earnings-leverage relationship.

Resumo

Título: A teoria do trade-off pode corresponder melhor às evidências empíricas ao implementar os ganhos reversos à média? Um estudo teórico sobre a teoria do trade-off

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Palavras-chave: Ótima estrutura de capital; Teoria do trade-off; Taxa de alavancagem

A teoria tradicional do trade-off prevê uma relação positiva entre o lucro e o rácio de alavancagem, contrariando a evidência empírica que sugere uma relação negativa entre os dois. O objetivo desta dissertação é analisar se a introdução de um processo de reversão para a média na teoria tradicional do trade-off torna os resultados do modelo mais alinhados com a evidência empírica.

Na dissertação, eu implemento o modelo de Sarkar e Zapatero (2003), que considera que o lucro da empresa antes de juros, impostos e depreciações reverte para a média em uma configuração semelhante à proposta por Leland (1994). Eu uso este modelo para calcular o rácio de alavancagem ótimo. Os parâmetros base apresentados por Sarkar e Zapatero (2003) foram considerados e obtiveram-se resultados iguais. O modelo foi resolvido numericamente para a barreira ótima de insolvência, nível ótimo de cupão e, finalmente, para o rácio ótimo de alavancagem. Além disso, consideraram-se diferentes níveis de lucros iniciais, diferentes velocidades de reversão para a média e diferentes níveis de lucros médios para analisar como o comportamento do modelo perante diferentes parâmetros.

Conforme argumentado por Sarkar e Zapatero (2003), a consideração da reversão para a média conduz a uma relação negativa entre o rácio ótimo de alavancagem e os lucros, o que está de acordo com a evidência empírica. Este resultado contribui para mudar a ideia generalizada de que a teoria do trade-off não é consistente com a realidade relativamente à relação lucro-alavancagem.

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1. Introduction

Since Miller and Modigliani proposed their irrelevance propositions in 1958, the search for the optimal capital structure is considered one of the biggest and most important questions in corporate finance. The issue is grounded in the continuous quest of maximising the wealth of equity holders and has spiked questions regarding how different capital structures can contribute with creating value for the firm, and as such, their equity holders. The issue has seen increasing focus as new theories and models arise and are explored.

As theories were explored, two major competing models arose. The traditional trade-off theory and the newer pecking order theory. The trade-off theory builds its model on the trade-off between the tax benefits of debt and the costs associated with such debt, notably, bankruptcy costs or financial distress costs. According to this theory there is an optimal level of debt and several models have been proposed in order to determine its value. The pecking order theory on the other hand, does not suggest the existence of an optimal capital structure. Instead, it provides a prioritized capital use order. According to this theory firms try to obtain funding in the following order: cash (retained earnings), guaranteed debt (safe debt), risky debt and lastly new equity. The latter theory has been seen as superior due to its suggestion of a negative relation between earnings and leverage ratio. In turn, the traditional trade-off theory suggests a positive relationship between these, which is not supported by empirical evidence. Thus, the pecking order theory has been prioritized, due to its support by empirical evidence.

In 1978 Bhattacharya introduced mean-reverting earnings to the trade-off theory. It was argued that it is more in line with the data that earnings revert to a long term mean rather than wander forever. This follows the basic economical principle that as long as earnings is maintained at an increased level, competitors would enter the market and start competing with the firm. Consequently, pushing profits downward. Similarly, as earnings drop, competitors would leave the market either by bankruptcy or choice, and thus limiting new supply, in line increasing earnings for the remaining firms. This approach has been further developed and explored by Raymar, who states that recent evidence “suggests that earnings processes are heterogeneous and exhibit mean-reverting tendencies.” (Raymar, 1991).

In this dissertation I implement the mean reverting model proposed by Sarkar and Zapatero, which extends Raymar and Bhattacharya models to a setting similar to the one proposed by Leland in 1994. Under this approach, the firm defaults when equity holders find no value in keeping the firm afloat, and as such, all potential value (minus the default costs) goes to

debtholders. I show that including mean reverting earnings shifts the model conclusions to support the empirical evidence of a negative relation between earnings and leverage. I illustrate this through a comparative statics exercise.

The dissertation follows the introduction with the following structure. Chapter 2 – Literature Review, where I discuss previous theory and applications in the discussions of the optimal capital structure and its significance. Chapter 3 – Model, where the input variables and formulas of the trade-off theory with mean-reversion is discussed and shown. Chapter 4 – Numerical Analysis, where I run a comparative statics exercise. Chapter 5 – Conclusion.

2. Literature Review

2.1 Structural models of contingent claims pricing and credit risk

Originating in 1958, with the irrelevance proposition of Modigliani and Miller, the search for the optimal capital structure is one of the most impactful and most explored issues in corporate finance. Modigliani and Miller propose several propositions. In their seminal 1958 paper, they stated that under certain assumptions, the difference in value between a firm financed with equity versus a firm financed with debt is equal to zero. I.e. the financing of a firm does not affect the value of that firm. (Modigliani & Miller, 1958)

In 1963, only five years after their first irrelevance proposition, Modigliani and Miller published a correction to their paper, answering the critics of their irrelevance proposition. The updated paper takes the tax benefit associated with debt as certain and discounts it at the risk-free rate. In this case, they show that, differently to what they originally pointed out, that one firm with twice the expected return after tax does not necessarily have twice the actual return after tax. As a result, there is no arbitrage process that can lead two firms with different degrees of leverage to the same value. Though Modigliani and Miller's "corrected" model leads to the conclusion that the firm should use the maximum amount of debt possible, the authors point to the reasoning that the firm's optimal leverage ratio should result from balancing the marginal benefits and the marginal costs associated with debt, most of them being out of their static model framework.

In 1973, Kraus and Litzenberger considered "corporate taxes and bankruptcy penalties into a single-period valuation model in a complete capital market.". They argue that in this setting "the total market value of the firm is *not* in general a concave function of financial leverage." as one might think. (Kraus & Litzenberger, 1973)

Three years after Kraus and Litzenberger's work, in 1976, Scott proposed a multiperiod model to determine the optimal capital structure. His model, presented in the paper *A Theory of Optimal Capital Structure*, accounts to tax benefit associated with debt as well as the costs of bankruptcy. (Scott, 1976) Under plausible conditions his model implies a unique optimal structure. The model proposed, undertakes the trade-off between tax advantages and the distress costs (bankruptcy costs) that emerge with leverage, as the possibility for distress increases. The firm's financing mix determines the firm's debt obligation and the attributable tax savings to debt financing, thus the tax gains. The trade-off occurs as the increased debt obligation increases the possibility of bankruptcy and thus the incurring penalties attributed to the bankruptcy.

However, as in Kraus and Litzenberger (1973), the firm's market value is not necessarily a concave function of the firm's debt obligation. This applies in some states if the firm's earnings are exceeded by the firm's debt obligation. (Kraus & Litzenberger, 1973)

Brennan and Schwartz proposed a dynamic trade-off model in continuous time in 1984. Their paper developed the equilibrium valuation model by Cox, Ingersoll and Ross in their unpublished paper in 1978 (published in 1985). (Cox, Ingersoll, & Ross, 2005 (1985)) Brennan and Schwartz simplified the model projected by Cox, Ingersoll and Ross considerably to make it analytically usable. This was done by assuming, among other assumptions, a constant proportional risk aversion. This assumption ensures that the coefficient σ_{iw} (which here represents the covariance of ROE on security i with the return on aggregate wealth) remains positive. This paper further implements a first step towards the analysis of dynamic financing policies, by suggesting that the optimal policy includes at least three facets. 1. An optimal indenture. 2. An optimal initial capital structure and 3. The optimal financing policy, given the current capital structure. Whereas the last part includes a dynamic aspect, which up to this point had been largely neglected in studies. (Brennan & Schwartz, 1984)

Also in 1984, Kane Marcus and McDonald published their take on the capital structure theory, where they aimed to find the reasoning behind the observed debt-to-firm-value ratios in the U.S. economy. These authors develop a valuation model for levered firms, which includes bankruptcy costs and incorporates personal taxes. The paper suggests, that as the tax advantage of debt is decreased, it is less costly to deviate from the optimal debt-to-firm-value. Similarly, as the tax advantage of debt is increased, it is increasingly costly to deviate. Furthermore, they find that the advantage of debt financing should be "measured as a rate of return per period". Thus, as the personal tax rate moves towards the corporate tax rate, the advantage to debt measured as a rate of return is small, and therefore, the trade-off between tax advantages and bankruptcy costs are unlikely to largely explain the different observed debt-to-firm-value ratios. (Kane, Marcus, & McDonald, 1984)

Observing another approach to the optimal capital structure, we observe the approach of Leland 1994, the model which Sarkar and Zapatero 2003 mostly uses as a foundation for their paper. Leland's model's focus is not the optimal leverage ratio directly, but it is used to calculate the optimal coupon payment and thus, the optimal leverage ratio. Leland assumes lognormal earnings, instead of the mean-reverting earnings of Raymar. As another change towards the approach of Raymar, Leland assumes that the face value of debt is not readjusted and is

therefore a one-time decision that cannot be changed. Further, Leland assumes that the debt issued is long-term debt, more in line with what observed in the market. Leland concludes that the optimal coupon payment is proportional to firm value, but that the optimal leverage ratio itself is independent of such value. (Sarkar & Zapatero, 2003) (Leland, 1994)

The assumptions of a constant debt ratio, and thus a stable trade-off between tax advantages and the bankruptcy costs is dominant in the earlier applications of the trade-off theory. Firm values are expectedly not stable, and firms should look for an optimal trade-off in a dynamic world. D. Duffie's book published in 1992, proposes a formula used by Sarkar and Zapatero to use expected PV's (present values) for valuation of cash flow streams or securities, irrespective of the preferences of investors. (Duffie, 1992) The dynamic models were also discussed earlier, among others by Brennan and Schwartz. In 2001, Goldstein, Ju and Leland proposed their EBIT-based model of dynamic capital structure. They raise questions and criticism on earlier studies as the one performed by Brennan & Schwartz in 1978 (1984), citing that the risky debt is significantly riskier than what is modelled in such traditional models because firms may issue further debt leading to an increase in the bankruptcy threshold over time. Goldstein, Ju and Leland extend the Black and Scholes proposed (1973) approach, where total capital (equity + debt) was seen as being lognormally distributed. (Black & Scholes, 1973) In their case, Goldstein, Ju and Leland take EBIT (earnings before interest and taxes) as log normal. It is assumed that EBIT is independent of changes in the firm's capital structure, which follows the much-used assumption that investment and financing policy is separated. The most distinctive feature of their paper is that they consider that the firm has the possibility to recall current debt and issue more debt whenever the firm value hits a certain threshold. They argue that accounting for increasing debt, or upward restructuring may in practice be more important than accounting for decreasing debt, or downward restructuring. (Goldstein, Ju, & Leland, 2001)

The trade-off theory of capital structure is not the only capital structure theory. Almost a decade later, after the trade-off theory has been proposed, Myers proposed the second of the major competing models of the optimal leverage puzzle in his 1984 paper, *The Capital Structure Puzzle*. (Myers, 1984) Called the pecking order theory, Myers's hypothesis is based on the information asymmetry between managers or shareholders (insiders) and outsiders. According to Myers, firms prefer to pursue financing with internal finances first. Secondly, the firm will obtain financing through covered debt (or secured debt with coherent guarantees). Thirdly, raising funds through risky debt and finally, raising funds through issuing equity. This is grounded in the empirical conclusion that suggests that firms have good reasons to not finance

by issuing equity or risky securities. Nevertheless, the solution proposed by the pecking order theory does not propose an optimal capital structure but propose the optimal *order* in which capital should be used. The other possibilities thus tackle the dilemma where the firm do not want to pass on positive-NPV projects as well as not wanting to issue stock at prices lower than the value they believe to be right. (Myers, 1984)

The pecking order theory has received empirical support, and in a head-to-head comparison between the pecking order theory and the trade-off theory, the pecking order seems to be viewed more favourably. One of the reasons for this is the trade-off theory suggestion of a positive relationship between leverage and earnings. (Brealey & Myers, 1996) (Shyam-Sunder & Myers, 1999) In contrast, the pecking order hypothesis predicts this relationship to be negative in line with empirical evidence. More recent papers have nevertheless questioned whether the trade-off theory really implies a positive relationship between leverage and earnings. Among these, Sarkar and Zapatero chose to re-examine the trade-off theory when corporate earnings follow a mean-reversion process. These authors show that in this case their trade-off model leads to a negative relationship between earnings and leverage. (Sarkar & Zapatero, 2003)

Mean-reversion models are not new in the literature. Starting with Bhattacharya in 1978, several papers have argued in favour of using mean reverting processes. Bhattacharya looks to the capital budgeting problem that started with Myers and Turnbull's paper of 1977. Myers and Turnbull used cash flow stochastic processes to investigate the "accuracy of traditional methods of valuation like discounting at a risk-adjusted cost of capital". (Bhattacharya, 1978, p. 1317) Following Hamada's suggestion in 1977, Bhattacharya argues that mean-reverting cash flow processes are likely to be more economically relevant than the processes provided by Myers and Turnbull as well as Treynor and Black. The rationale used by Bhattacharya is that in a competitive economy, the expectation is for "project cash flows to revert to levels that make firms indifferent about new investments in the particular type of investment opportunity that a given project represents rather than "wandering" "forever.". (Bhattacharya, 1978, p. 1317) Further, mean reverting earnings have been empirically justified in several studies. Ranging from Raymar's 1991 statement "Recent evidence, however, suggests that earnings processes are heterogenous and exhibit mean- reverting tendencies" to the 1987 paper by Kormendi and Lipe. (Raymar, 1991, p. 327) (Kormendi & Lipe, 1987) (Myers & Turnbull, 1977) (Hamada, 1977) (Treynor & Black, 1976)

While earlier papers have modelled cash flows using stochastic processes, Raymar's paper, published in 1991, introduces changes to the original trade-off theory within the capital structure decision. Raymar introduced and included mean reverting earnings into the model, where earnings (in the specific model EBIT) at date t followed a mean-reverting first order autoregressive process.

$$X_t = (1 - \phi)\mu + \phi X_{t-1} + \epsilon_t \equiv \mu_t + \epsilon_t$$

Raymar considers a multiperiod capital structure model. Earnings are assumed to be mean-reverting and cash flows are assumed to follow an autoregressive process, where $(1 - \phi)$ measures the speed of reversion in a stationary Markov process. In his paper, Raymar argues that "If time series attributes of earnings affect the variability and permanence of a firm's cash flow streams, they may, in turn, influence optimal leverage decisions." (Raymar, 1991, p. 328) Raymar calculates the optimal leverage ratio for the firm assuming that the firm only issues short-term debt. Further, the assumption of readjusting debt levels at a cost equal to zero is used. According to his model, during an increase (decrease) in earnings the optimal leverage ratio increases (decreases). (Raymar, 1991)

3. Sarkar and Zapatero (2003) model

This section presents the mean-reverting structural model proposed by Sarkar and Zapatero.

3.1 Model setup

Sarkar and Zapatero consider a company that holds a perpetual project, financed by Equity (E) and a single class of perpetual Debt (D) with a constant coupon C. The debt level is assumed fixed until bankruptcy. The project earnings, projected by EBITDA (Earnings Before Interest, Taxes, Depreciation, and Amortization), are given by the following somewhat modified arithmetic Ornstein-Uhlenbeck process:

$$dx_t = k(\theta - x)dt + \sigma x dW_t^Q \quad (1)$$

In the formula, k is the mean reversion speed, θ is the long-term mean earnings which x reverts to, and σ controls the volatility of the projects returns. This is a mean reverting process in all cases except the ones where $k = 0$ or $\theta = 0$. In these cases, earnings (EBITDA) become lognormal with drift 0 and $-k$. Under this formula EBITDA can never be ≤ 0 as it is a reflecting boundary.

The process implemented by Sarkar and Zapatero is also recognized as the inhomogeneous geometric Brownian motion (IGBM) because the expected return of the state variable is not homogeneous. The inhomogeneous GBM (IGBM) varies from the GBM in other aspects. In particular, the IGBM is a stationary process while the GBM is not.

The model further assumes that if the EBITDA is lower than the coupon C, shareholders must inject capital, as proposed by Leland in 1994. This follows the approach that if equity has “some value, a firm will always be motivated and able to issue additional equity to cover any shortage, rather than default”. (Sarkar & Zapatero, 2003) If capital is not injected, the firm defaults on its debt. Expectedly, shareholders choose the level x_L , marking the optimal default level. Given default, shareholders receive nothing, and debtholders receive $(1 - \alpha)$, where α refers to bankruptcy costs (given in percent of firm value, thus $0 \leq \alpha \leq 1$). (Leland, 1994)

It is assumed that the firm pays corporate taxes and that shareholders pay dividend taxes. The constant effective tax rate is noted as τ . As other trade-off capital structure models, the firm issues debt because it has a tax advantage. If C is larger than EBITDA, the firm earns $\tau(C - EBITDA)$, thus assuming no loss on negative after-interest earnings tax benefits. Tax advantages are assumed lost in the case of default, and the risk-free interest rate r is assumed

constant over the projects timeline, consistent with previous practices by Leland in 1994. (Leland, 1994)

3.2 Unlevered firm value

As the firm holds only one project, its value is equal to that of the project. Furthermore, as earnings are not affected by different capital structures, the firm value can be simply determined as the expected PV (present value) of the after-tax perpetual earnings (EBITDA) stream. Under no arbitrage conditions, this is given by

$$V_t = V_1 x + V_2 \quad (2)$$

Where $V_1 = \frac{1-\tau}{r+k}$; $V_2 = \frac{(1-\tau)k\theta}{r(r+k)}$. By substituting on the expression, we obtain.

$$V_t = \frac{1-\tau}{r+k} x + \frac{(1-\tau)k\theta}{r(r+k)} \quad (3)$$

Firm value is composed by a transitory and a permanent component. Changes in earnings (x) only affects the value through the transitory part of the formula. V_1 captures how firm value changes with earnings. The speed reversion parameter k affect each of these components differently. V_1 is negatively affected by a rising k and V_2 is positively affected by the change. In the extreme scenarios where earnings move towards 0 and/or k moves towards infinite, V_1 , and thus the transitory part, move towards 0. In those cases, the firm value converges towards $V(x) = V_2$.

From equation 1 and 2, the dynamics of the firm value can be further elaborated using Ito's lemma:

$$dV_t = k[(V_1\theta + V_2) - V_t]dt + \sigma x V_1 dW_t^Q \quad (3)$$

The dynamics of firm value thus inherits the mean-reverting property of the underlying earnings cash flow x . Furthermore, the firm value converges to its long-term value, equaling the projected value evaluated at θ .

The variance for the earnings stream is given by Equation 4:

$$Var\left(\frac{dx}{x}\right) = \sigma^2 dt \quad (4)$$

For the firm value, the variance is given as

$$Var\left(\frac{dV}{V}\right) = Var\left(\frac{\sigma x V_1}{V} dW_t^Q\right) = \left(\frac{\sigma x V_1}{V}\right)^2 dt \quad (5)$$

Which can be simplified to

$$Var\left(\frac{dV}{V}\right) = \left(\frac{\sigma}{1 + \left(\frac{k\theta}{rx}\right)}\right)^2 dt \quad (6)$$

As both k, r, x and θ are positive, the volatility of the firm value is lower than that of the earnings cash flow of the project itself. If the mean reversion was absent (i.e. $k = 0$) the project becomes equal to a GBM, in which case the rate of change of earnings and firm value have the same volatility. Furthermore, we see that as the mean reversion increases, the firm risk decreases, and as $k \rightarrow \infty$ the risk disappears in its entirety.

3.3 Debt pricing

Following standard no-arbitrage arguments, debt value must satisfy the ordinary differential equation:

$$k(\theta - x) \frac{\partial D}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 D}{\partial x^2} + C = rD \quad (7)$$

whose general solution is given by

$$D(x) = \frac{C}{r} + D_1 x^{\gamma_1} M_1(x) + D_2 x^{\gamma_2} M_2(x) \quad (8)$$

The parameters γ_1 and γ_2 are the positive and negative roots of the quadratic equation

$$\frac{1}{2} \sigma^2 \gamma(\gamma - 1) - k\gamma - r = 0 \quad (9)$$

$M_1(x)$ and $M_2(x)$ are given by equation 10:

$$M_1(x) = M\left(-\gamma_1; 2 - 2\gamma_1 + \frac{2k}{\sigma^2}; 2k \frac{\theta}{\sigma^2 x}\right) \quad (10)$$

$$M_2(x) = M\left(-\gamma_2; 2 - 2\gamma_2 + \frac{2k}{\sigma^2}; 2k \frac{\theta}{\sigma^2 x}\right)$$

where M is the Kummer confluent hypergeometric function.

As x grows, default risk declines. When x reaches infinity, default risk is 0. To prevent bond values from “blowing up” as bond value approaches $\frac{C}{r}$ we impose the usual boundary conditions. (Sarkar & Zapatero, 2003)

$$\begin{cases} \lim_{x \rightarrow +\infty} D(x) = \frac{C}{r} \\ D(x_L) = (1 - \alpha)(V_1 x_L + V_2) \end{cases} \quad (11)$$

This leads to

$$\begin{aligned} D_1 &= 0 \\ D_2 &= \frac{(1 - \alpha)(V_1 x_L + V_2) - \left(\frac{C}{r}\right)}{(x_L)^{\gamma_2} M_2(x_L)} \end{aligned} \quad (12)$$

By inserting these results into formula 8 we obtain the debt value

$$D(x) = \frac{C}{r} + \frac{(1 - \alpha)(V_1 x_L + V_2) - \left(\frac{C}{r}\right)}{(x_L)^{\gamma_2} M_2(x_L)} x^{\gamma_2} M_2(x) \quad (13)$$

By further simplifying we attain:

$$D(x) = \frac{C}{r} + \left[(1 - \alpha)(V_1 x_L + V_2) - \frac{C}{r} \right] \frac{M(x)}{M(x_L)} \left(\frac{x}{x_L} \right)^{\gamma_2} \quad (14)$$

Alternatively, the value of debt can be written as

$$D(x) = \frac{C}{r} [1 - p_B] + (1 - \alpha)(V_1 x_L + V_2) p_B \quad (15)$$

Equation 15 is an extension to the bond valuation presented by Leland in 1994 and includes mean-reverting earnings. The second term of the equation is negative, which can be intuitively explained, as the value of debt decreases when we include distress costs. Thus, the value of $(1 - \alpha)(V_1 x_L + V_2)$ must be lower than $\frac{C}{r}$, making the term negative. The first term represents the value of debt in the absence of distress costs.

Consequently, the second term represents the recovered value in the case of bankruptcy, given by $(1 - \alpha)$ times the firm value. In the alternative equation (equation 15), $\frac{M(x)}{M(x_L)} \left(\frac{x}{x_L}\right)^{\gamma_2}$ is substituted with p_B . Thus, $p_B = \frac{M(x)}{M(x_L)} \left(\frac{x}{x_L}\right)^{\gamma_2}$.

3.4 The value of the tax benefit

We value the tax benefit as if it was a security denoted by TB. Similar to debt, if the firm had no default risk, which occurs when earnings go to infinity, the value of the tax benefit would be equal to $\frac{\tau C}{r}$. If the firm defaults, the tax benefit disappears in its entirety. Setting up a second order ODE similar to the one in equation 7 with τC instead of C and with boundary conditions as follows.

$$\begin{cases} \lim_{x \rightarrow +\infty} TB(x) = \frac{\tau C}{r}, \\ TB(x_L) = 0 \end{cases} \quad (16)$$

we find the value of the tax benefit to be

$$TB(x) = \frac{\tau C}{r} + \left[0 - \left(\frac{\tau C}{r}\right)\right] p_B = \frac{\tau C}{r} [1 - p_B] \quad (17)$$

3.5 The value of distress costs

Following the same process as for tax benefits, the firm occurs a bankruptcy cost in the case of bankruptcy, which needs to be subtracted of the firm value of the unlevered firm. The cost is calculated as a security that pays nothing in the case of no bankruptcy and has the value of α times the value of the unlevered firm at default. Furthermore, the distress cost value approaches 0 as earnings increases and the possibility of default decreases. These restrictions correspond to the following boundary conditions:

$$\begin{cases} \lim_{x \rightarrow +\infty} DC(x) = 0 \\ D(x_L) = \alpha(V_1x_L + V_2) \end{cases} \quad (18)$$

We find the value of the distress cost to be

$$DC(x) = 0 + [\alpha(V_1x_L + V_2) - 0]p_B = \alpha(V_1x_L + V_2)p_B \quad (19)$$

3.6 Total firm value and the value of equity

As put forward in the paper by Leland in 1994, the total firm value ($FV(x)$) is calculated by the unlevered firm value plus the tax benefits less the distress costs.

$$FV(x) = V_1x + V_2 + TB - DC \quad (20)$$

Substituting TB and DC in equation 20,

$$FV(x) = V_1x + V_2 + \frac{\tau C}{r} [1 - p_B] - \alpha(V_1x_L + V_2)p_B \quad (21)$$

Equity value ($E(x)$) is given by subtracting the value of the debt from the total firm value. This provides us with the following equation for equity value.

$$\begin{aligned} E(x) &= FV(x) - D(x) \\ &= \\ &V_1x + V_2 + \frac{\tau C}{r} [1 - p_B] - \alpha(V_1x_L + V_2)p_B \\ &\quad - \frac{C}{r} [1 - p_B] - (1 - \alpha)(V_1x_L + V_2)p_B \end{aligned} \quad (22)$$

This can be simplified to

$$E(x) = V_1x + V_2 - \frac{C}{r} (1 - p_B)(1 - \tau) - (V_1x_L + V_2)p_B \quad (23)$$

The value of equity is further limited by the earlier provided rationale that if the firm defaults, the equity is lost. Thus, at $x = x_L$; $E(x_L) = 0$.

3.7 The optimal default boundary

As mentioned previously, in the case of long-term debt an endogenous default trigger is appropriate. Thus, in the model proposed by Sarkar and Zapatero, equity holders choose such a default trigger themselves, optimally, to maximize wealth. They will choose the default trigger by either injecting additional equity to keep the firm afloat, or not. In the case where equity holders find it not sustainable to inject more capital, default will occur. The equity value can then be put forward as such. $x_L = \text{argmax } E(x)$. Furthermore, the optimal default boundary will be largely dependent on the coupon C . As the coupon grows, the cost of debt and thus the amount of inherent cost the equity holder must bear increases, and default will intuitively occur sooner. Likewise, if the coupon decreases, or is very small, the inherent cost of keeping the firm afloat decreases, the default will occur later, and the default boundary will decrease.

Since the default trigger is largely dependent on the coupon, the optimal trigger is known for each level of C , and is a part of the debt apprehending decision. Consequently, equity holders will choose the level of C to maximize firm value at $t = 0$. This decision is made, given the rational put forward earlier. The optimal default boundary x_L will thus be determined by the smooth pasting condition, where equity is zero at default, thus derivatizing $E(x)$ on x to make $x = 0$ at $x_L = x$. This can be shown as

$$\left. \frac{\partial E}{\partial x} \right|_{x=x_L} = 0 \quad (24)$$

By solving the equation $E(x)$ with respect to x , we get

$$\frac{M \left(1 - \gamma_2; 3 - 2\gamma_2 + 2 \frac{k}{\sigma^2}; 2k \frac{\theta}{\sigma^2 x_L} \right)}{M \left(-\gamma_2; 2 - 2\gamma_2 + 2 \frac{k}{\sigma^2}; 2k \frac{\theta}{\sigma^2 x_L} \right)} + \frac{x_L \gamma_2 - \left[\frac{V_1(x_L)^2}{V_1 x_L + V_2 - (1 - \tau) \frac{C}{r}} \right]}{\frac{\gamma_2 k \theta}{(1 - \gamma_2) \sigma^2 + k}} = 0 \quad (25)$$

As there are no analytical solution to the equation, the solution, the optimal default boundary, must be found numerically. As in the original paper, I will use the “base case” values proposed. The proposed values are $r = 0.07$, $\alpha = 0.5$, $\tau = 0.15$, $k = 0.1$, $\theta = 1$, $\sigma = 0.4$ and $C = 1$.

With these numerical input values, the optimal default boundary is $x_L = 0.4389$. Thus, it is profitable for equity holders to keep the firm afloat by inserting more equity or issue more equity if earnings are above x_L . They need to do this as earnings are lower than the coupon C .

Even though equity holders need to cover the negative cashflow as x falls below C , they expect earnings to come back up, since earnings are mean reverting with mean earnings of 1 ($\theta = 1$). Furthermore, this signalizes a still positive expected present value in the firm. As the present value for equity holders reaches zero, it is optimal to default, being the optimal default boundary x_L .

3.8 The determinants of debt value when shareholders set the default boundary

Replacing the barrier value in equation 14 by the solution to equation 25, one obtains debt value considering shareholders' optimal default barrier. This functions as an extension to the debt valuation result proposed by Leland's 1994 paper for the case where mean reversion is considered. There are several findings by this formula, as debt valuation is found to (1) decrease as r, α and τ increases, and (2) increase as x, θ and k increases. This makes sense also intuitively, as higher risk-free rate would increase the alternative cost of financing and by such increase debt value. Furthermore, increased taxes would directly increase the tax advantage of attaining debt, and by such increase the direct value of debt.

Although debt value normally is a decreasing function of volatility σ , this is not correct for low values of x (i.e., $x \rightarrow x_L$). When earnings approach the optimal default boundary debt valuation is an increasing function of volatility. This increase in volatility postpones default or reduces the default boundary and consequently increases debt valuation. Similar observations may also be made on the coupon C , as debt valuation is an increasing function of coupon initially, but this changes as the coupon grows past C^* . Consequently, debt value is a decreasing function of coupon as the coupon grows. This can be explained by the 2 parts of equation 14. As the coupon increases the second part of the equation dominates, resulting in a higher default boundary and earlier default, consequently reducing debt value.

4. Numerical analysis

4.1 Base case

In this section of the dissertation, changes of the input parameters will be discussed. Consequently, the impact these changes has on the optimal leverage ratio and other output variables will be considered and explored. Results will be presented under each parameter and discussed as such. To look at the implications these changes have, I will first establish a base case, which will function as the base for changes. The base case parameters correspond to the ones presented in Sarkar and Zapatero (2003):

$$x = 1$$

$$\theta = 1$$

$$k = 0.1$$

$$\sigma = 0.4$$

$$r = 0.07$$

$$t = 0.15$$

$$\alpha = 0.5$$

By inputting these values, solving for the optimal default barrier and solving for the optimal coupon, we attain the following values for the base case:

Leverage ratio of 0.6636

Default barrier of 0.0856

Optimal coupon of 0.6267

Debt value of 8.9438

Equity value of 4.5335

Firm value of 13.4772

Debt yield of 7.01%

4.2 Debt capacity

The maximum debt value can be seen as the maximum debt capacity for the firm, as it is the maximum amount of debt attainable. The value of debt, or the debt capacity, can be seen as an increasing function of the coupon C as the debt value is below its max. As the function is concave, the debt value decreases with all increases of the coupon after the point at which it has hit its maximum value, say D_{max} . Given the nature of such a concave function, the value of debt does have one maximum value for a given coupon C .

As x_L is dependent on the coupon, the debt capacity must be found numerically. I have continued with the assumed parameters earlier exemplified, and with the earnings x_0 normalized to 1. The debt capacity function for two estimates of mean reversion is shown in *Figure 1*, where $k = 0.1$ and $k = 0.2$ are the input values for the mean reversions. As can be observed from the figure, both functions operate in a concave motion as expected, and reaches a maximum value for some specific value of C .

The maximum debt capacity reached with $k = 0.1$ is 9.103, giving $D_{max} = 9.103$ at $C_M = 0.65$. As a result, the coupon rate of the debt is $\frac{0.65}{9.103} = 0.0714 = 7.14\%$. Furthermore, with a firm value calculated by equation 20, that gives us a leverage ratio of 68 %. *Figure 1* also includes the debt value in the case of higher mean reversion ($k = 0.2$). With all other inputs unchanged the increase in mean reversion increases the debt capacity to 11.1966 with a C_M of 0.788. This gives a coupon rate of 7.04 % and a leverage ratio of 81%.

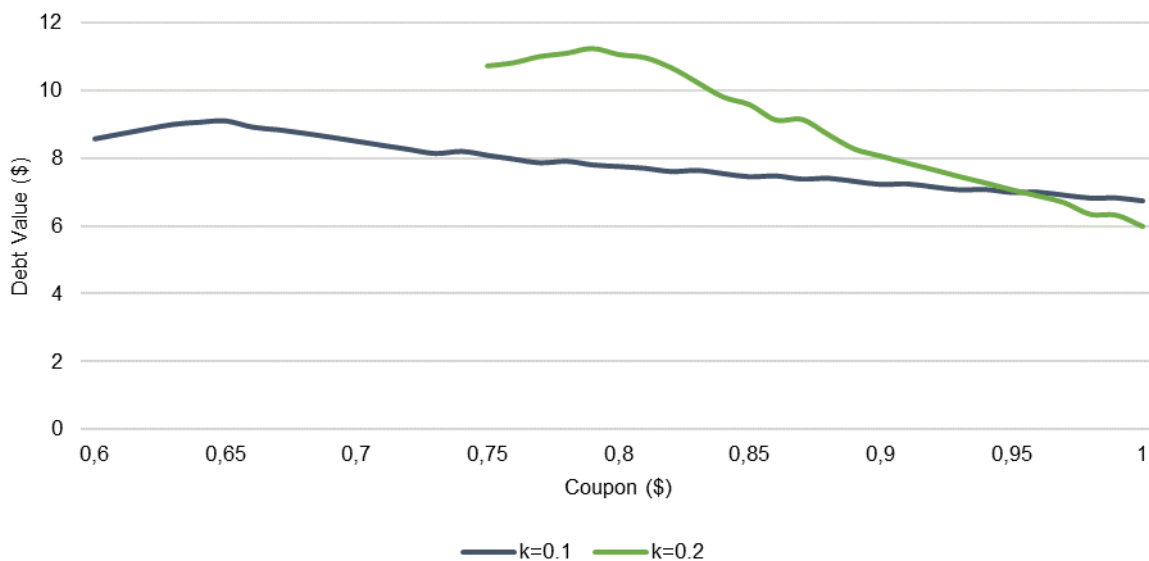


Figure 1 – Debt capacity

4.3 Optimal leverage ratio

The optimal coupon is the one that maximizes firm value. Figure 2 below shows the impact of coupons on firm value. Intuitively, and expectedly, the firm value is a concave function of the coupon value (i.e. firm value rises before it falls as the coupon is increased). The value that maximizes the firm value is computed using a numerical search. Under our base case parametrization, the optimal coupon is $C^* = 0.6267$. The corresponding default boundary is $x_L = 0.0856$. Taking these values, the value of the firm, debt, and equity are, respectively, $FV(x) = 13.4772$, $D(x) = 8.9438$ and $E(x) = 4.5335$. The optimal leverage ratio is thus 66.36%. As the unlevered firm value is $V_1 x_0 + V_2 = 12.1429$, the levered firm is worth approximately 11 % more than the unlevered firm. As a result, the addition of leverage to the firm or project can significantly increase firm value and thus the wealth of shareholders. The debt yield is 7.01 %. Given that the risk-free rate is 7%, this corresponds to a spread of 0.01%. Interestingly, as noted earlier, the optimal leverage should not be the same as the one for maximum debt value. In this case we observe an optimal coupon significantly lower than that seen in Figure 1, where we find the optimal debt capacity to be $D_{max} = 9.103$ with a coupon of $C_M = 0.65$. This is according to previous theoretical findings.

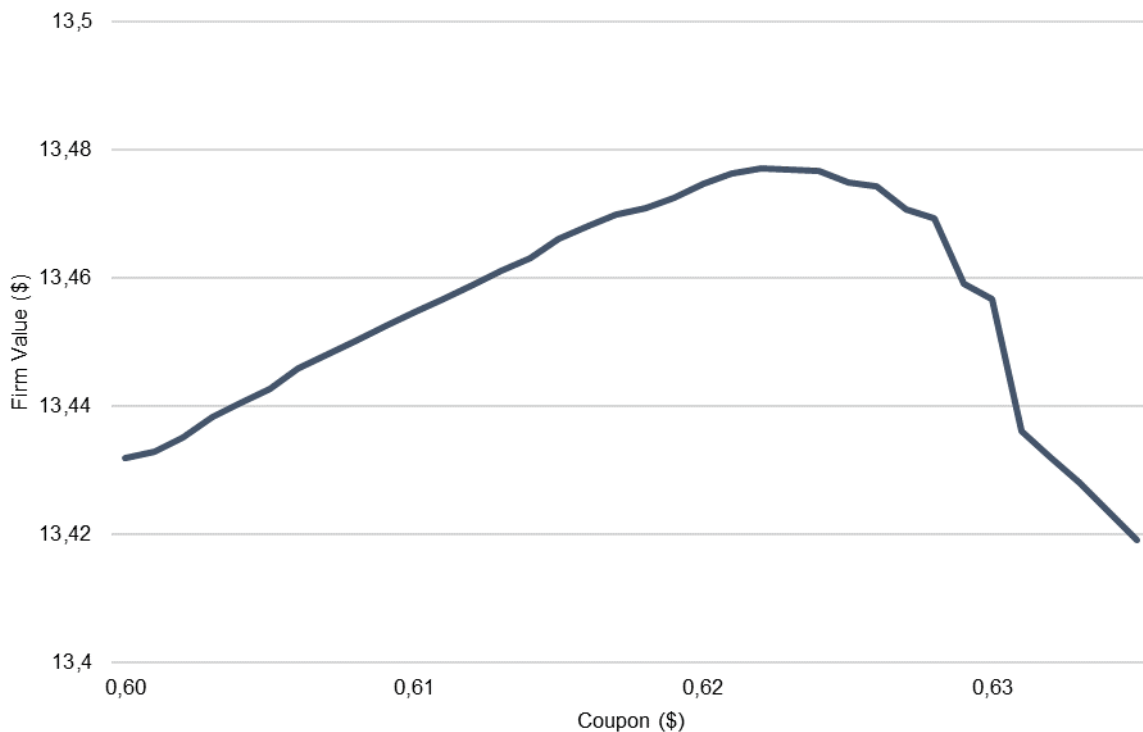


Figure 2 – Optimal leverage ratio

4.4 Comparative statistics

4.4.1 Earnings as the determinant

4.4.1.1 Impact on the optimal coupon

In *Table 1a* the optimal coupon level C is shown for the different variations of earnings and mean reversion. To test whether the model holds the assumption of constant coupon level, the optimal coupon should not change as the earnings x changes. If the optimal coupon changes linearly with changes in earnings, the model suffers from the same limitations as the one presented by Leland in 1994.

Table 1a
Optimal coupon level

x	$k = 0$	$k = 0.1$	$k = 0.2$	$k = 0.3$
0.5	0.1799	0.6260	0.7743	0.8488
1	0.3598	0.6267	0.7778	0.8488
1.5	0.5397	0.6270	0.7771	0.8488
2	0.7196	0.6273	0.7789	0.8488
2.5	0.8995	0.6276	0.7789	0.8488

Table 1-a - Optimal coupon level

As observed from *Table 1a*, without mean reversion (i.e. $k = 0$), the optimal coupon is directly proportionate to earnings. Consequently, a 100 % increase in x provides a 100 % increase in the optimal coupon level. This is in line with the findings of Leland 1994. The results are significantly different with mean reverting earnings (i.e. $k > 0$). For $k = 0.1$ we only observe a change in the optimal coupon from 0.6260 to 0.6276 for a 5-fold increase in earnings x (0.5 to 2.5). This corresponds to less than 1 %. With $k = 0.2$ we observe similar results with the optimal coupon level increasing again less than 1 %. Consequently, we can observe that the optimal coupon level is in practice close to independent from current earnings. This implies that for given parameter values, as included here, the coupon can be seen as almost constant. This result supports the assumption of constant coupon implied by the model. As the mean reversion increases, the coupon value tends to take an increasingly narrow band of values. This can be seen for $k = 0.3$, where the coupon is already constant for the 4th decimal.

Table 1b

Loss in value with a “no adjustment” policy

x	C	$D(x)$	$E(x)$	$FV(x)$	Leverage	Loss $FV(x)$
0.5	0.6260	8.9313	2.0420	10.9733	0.8139	
1*	0.6260	8.9356	4.5415	13.4771	0.6630	0.0007 %
1	0.6267	8.9438	4.5334	13.4772	0.6636	
1.5*	0.6260	8.9365	7.0415	15.9780	0,5593	0.0014 %
1.5	0.6270	8.9485	7.0297	15.9782	0.5600	
2*	0.6260	8.9369	9.5415	18.4783	0,4836	0.0024 %
2	0.6273	8.9527	9.5260	18.4788	0.4844	
	*Constant C version					

Table 1-b - Loss in value with a “no adjustment” policy

As proposed by Sarkar and Zapatero, we calculate the potential losses to firm value that arise from keeping a constant coupon value for different values of earnings. *Table 1b* takes as reference changes in values when $x = 0.5$. For all other input parameters, the values are consistent with the ones presented in the “base case”. In this case, the optimal coupon is 0.6260. We then consider how firm value increases with earnings. We do this by taking increments of 0.5. The total firm value, debt value, equity value and leverage are calculated for both the optimal coupon associated with each level of x and under the constant coupon in each case. The latter is denoted with an *. The last column in Table shows the loss to total firm value ($LossFV(x)$) from non-adjusting debt value in percentage of firm value. This value is the difference from the optimal readjustment case and the no adjustment case.

From *Table 1b* it is inherent that the loss in total firm value from implementing a constant coupon model is negligible, indicating a loss of 0.0024 % as current earnings quadruple from 0.5 to 2.0. Consequently, this supports earlier conclusions, supporting a constant coupon model. By observing *Table 1b* we see that by not “releveraging” as current earnings increase, the impact on the leverage of the firm is a slightly lower leverage-ratio. As observed with the loss in total firm value this change is negligible and only visible on the 3rd decimal point, and thus having limited to no impact.

4.4.1.2 Impact on the default barrier

Table 1c shows the impact changes in current earnings has on the optimal default barrier for equity holders. As observed in Table 1c, the default barrier is an increasing function of current earnings. Nevertheless, as earlier seen in section 3 and in line with Leland (1994), earnings are not directly impacting the equity holder's decision to default on the debt and thus initiating bankruptcy. The increase in the optimal default barrier is the result of the slight increase in optimal coupon that occurs whenever earnings increase. This also explains the consistent default barrier presented for $k = 0.3$, as the coupon in this case also is constant to the 4th decimal.

Sub-Conclusion: The default barrier is not affected by changes in earnings but is indirectly positively impacted by the positive relation between earnings and the optimal coupon.

Table 1c

Effects of increased earnings on the default barrier

x	$k = 0.1$	$k = 0.2$	$k = 0.3$
0.5	0.0842	0.1219	0.1874
1	0.0856	0.1333	0.1874
1.5	0.0862	0.1310	0.1874
2	0.0868	0.1369	0.1874
2.5	0.0874	0.1369	0.1874

Table 1-c - Effects of increased earnings on the default barrier

4.4.1.3 Impact on the optimal leverage ratio

Table 1d shows the impact current earnings has on the optimal leverage ratio of the firm. As shown in both Table 1b and Table 1d, the optimal leverage ratio is negatively impacted by increasing earnings if there is mean reversion (*i. e.* $k > 0$). In the absence of mean reversion, the optimal leverage ratio is fixed as the leverage ratio remains the same for different levels of earnings, as put forward by Leland (1994). Consequently, from Table 1b we can observe that as earnings increase, the value of equity ($E(x)$) increases significantly, but the increase in the value of debt ($D(x)$) is negligible. This is different from the traditional GBM model, where the optimal coupon increases significantly as earnings increase. In contrast, under the mean-reversion model, the optimal coupon increases very slightly and thus the leverage ratio decreases.

This is the practical explanation of the decreasing leverage ratio put forward in *Table 1d*. Thus, in the presence of mean reversion we observe the optimal leverage ratio to be a decreasing function of current earnings, implicating that an increase in current earnings decreases the leverage ratio. Further, as the mean reversion increases, the optimal leverage ratio increases. This is in line with previous conclusions.

Table 1d

Effects of increased earnings on optimal leverage

x	$k = 0$	$k = 0.1$	$k = 0.2$	$k = 0.3$
0.5	0.3025	0.8139	0.9046	0.9461
1	0.3025	0.6636	0.8045	0.8683
1.5	0.3025	0.5600	0.7215	0.8023
2	0.3025	0.4844	0.6559	0.7456
2.5	0.3025	0.4269	0.6002	0.6964

Table 1-d - Effects of increased earnings on optimal leverage

Sub-Conclusion: In the absence of mean-reverting earnings, the model corroborates earlier papers, such as the one by Leland in 1994, where the optimal leverage ratio does not decrease with current earnings. However, the inclusion of mean-reverting earnings turns the optimal leverage ratio a decreasing function of current earnings. The consideration of mean-reverting earnings is thus able to show that the trade-off theory of capital structure can produce a negative relation between the leverage ratio and earnings.

4.4.2 Mean-reversion as the determinant

4.4.2.1 Impact on the optimal coupon

Table 2a shows the impact mean-reversion has on the optimal coupon for the firm. The optimal coupon is largely affected by mean reversion, as mentioned later in the section of mean-reversions impact on the default barrier. By observing the values presented in *Table 2a* we see that this fits the practical results well for all levels of earnings, as long as mean reversion is present. At the absence of mean-reversion the optimal coupon is directly connected to earnings and will subsequently pass the optimal coupon levels including mean reversion. This result is easily seen in the $x = 2$ case, where the optimal coupon for no mean reversion is higher than that of $k = 0.1$. Nonetheless, in the presence of mean reversion, the optimal coupon is an increasing function of the levels of mean reversion. This can be explained by the earnings intuition previously described. The higher the mean reversion, the quicker earnings are expected to rise (or fall) to its long term mean θ , and thus implying reduced project risk.

Sub-Conclusion: The optimal coupon level is an increasing function of mean-reversion

Table 2a

Effects of mean-reversion on the optimal coupon

k	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0	0.1799	0.3598	0.5397	0.7196
0.1	0.6260	0.6267	0.6270	0.6273
0.2	0.7743	0.7778	0.7771	0.7789
0.3	0.8488	0.8488	0.8488	0.8488

Table 2-e - Effects of mean-reversion on the optimal coupon

4.4.2.2 Impact on the optimal leverage ratio

Table 2b shows the impact of changes in mean reversion on the optimal leverage ratio. As can be observed from Table 2b, the optimal leverage ratio is an increasing function of the mean reversion k . This is consistent with the results proposed by Raymar in 1991, and is consistent with what earlier proposed, as k implies reduced project risk. Furthermore, k represents an additional effect on the debt attractiveness, and thus, earnings apply a lower effect on the optimal leverage ratio as k increases. This can be explained as when earnings are high, and k is high, the earnings will revert quicker to its long term mean θ . Intuitively, this suggests that if $x < \theta$ and k are high, earnings will increase quicker. Oppositely, as $x > \theta$ and k are high, earnings will decrease faster. This implies that as k grows and $x > \theta$, k affects negatively. In Table 2b we also observe the impacts mean-reversion has on the optimal leverage ratio. In the absence of mean-reversion the optimal leverage ratio is constant, and on a significantly lower level than that of the mean-reversion cases. Nevertheless, as the speed of mean reversion increases the spread, or the impact of changes in current earnings decreases, consequently making changes in current earnings impactless if $k \rightarrow \infty$.

Sub-Conclusion: The optimal leverage ratio can be observed to be an increasing function of k . Furthermore, as the earnings increase, the optimal leverage is more sensitive to such change in the mean-reversion k . The increase in sensitivity can be explained by the quicker recovery to the long term mean that the increase in k represents. Thus, an anomaly from the mean (*i. e.* : $x \neq \theta$) would carry less weight as k is increased, and the optimal leverage ratio would by such be closer to that of the long term mean. Contrary, in the low mean reversion case, the optimal leverage ratio would decrease additionally.

Table 2b

Effects of mean-reversion on optimal leverage

k	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0	0.3025	0.3025	0.3025	0.3025
0.1	0.8139	0.6636	0.5600	0.4844
0.2	0.9046	0.8045	0.7215	0.6559
0.3	0.9461	0.8683	0.8023	0.7456

Table 2-f - Effects of mean-reversion on optimal leverage

4.4.2.3 Impact on the default barrier

Table 2c shows the impact of changes in mean reversion on the default barrier chosen by shareholders. As we observe in the table, as the speed of mean-reversion increases the default barrier increases substantially. This increase is mostly related to the inherent increase in the optimal coupon level in this case. This was previously discussed and shown in *Table 2a* and as observed for lower values of earnings in *Table 1c*.

Sub-Conclusion: The default barrier is an increasing function of mean-reversion, shown by the values in *Table 2c*. Noteworthy, this increasing function is mostly coherently connected to the increase in optimal coupon level.

Table 2c

Effects of mean-reversion on the default barrier

k	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.1	0.0842	0.0856	0.0862	0.0868
0.2	0.1219	0.1333	0.1310	0.1369
0.3	0.1874	0.1874	0.1874	0.1874

Table 2-g - Effects of mean-reversion on the default barrier

4.4.3 Mean-earnings as the determinant

Up to this point the mean earnings parameter θ has been taken as equal to 1. In this section I will look to the impact of changing this parameter.

4.4.3.1 Impact on the optimal coupon

Table 3a shows the mean-earnings impact on the optimal coupon. As expected, given the assumption of mean reversion and thus constant coupon rates, the optimal coupon is directly affected by mean-earnings. As can be seen from the table, an increase of 100% in mean-earnings increases the optimal coupon level by approximately 100%. As mean-earnings increase from the lowest estimation of 0.5 to the highest of 1.5, a threefold increase, the optimal coupon rises (for $x = 1$) to approximately threefold. This can be explained quite intuitively, as when the mean-earnings increase, everything else equal, the optimal coupon must also increase equally to keep project risk and other factors constant.

Sub-Conclusion: The optimal coupon is an increasing function of mean-earnings.

Table 3a

Effects of mean-earnings on the optimal coupon

θ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.5	0.3143	0.3135	0.3082	0.3082
1	0.6260	0.6267	0.6270	0.6273
1.5	0.9341	0.9365	0.9396	0.9396

Table 3-h - Effects of mean-earnings on the optimal coupon

4.4.3.2 Impact on the optimal leverage ratio

Table 3b shows the impact changes in mean earnings has on the optimal leverage ratio of the firm. Unsurprisingly, the mean-earnings θ has great implications on the optimal leverage ratio. The mean-earnings greatly impact V_2 (i.e. the permanent part of the project value), M_1 and M_2 . As can be observed in Table 3b, as mean-earnings increase, the optimal leverage ratio also increases. This applies for all levels of earnings. Also, as earnings increase, the optimal leverage ratio decreases. Interestingly, as earnings increase and the optimal leverage ratio decreases, the impact of increased mean-earnings tends to increase. Notice that the higher the spread between earnings and mean-earnings the more is earnings expected to fall/rise. Consequently, as earnings wander further away from its long term mean, they are expected to revert to that long term mean by the rate of reversion. Thus, as the difference increases the inherent change needed to revert to the mean increases. As this change increases, the optimal leverage ratio must also

change as the optimal leverage is inherently dependent on the long term mean. This can be explained by looking diagonally on the table, as we observe that as mean-earnings θ equal current earnings x , the optimal leverage ratios are approximately similar.

Sub-Conclusion: The optimal leverage ratio is an increasing function of mean-earnings.

Table 3b
Effects of mean-earnings on the optimal leverage ratio

θ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.5	0.6654	0.4843	0.3752	0.3093
1	0.8139	0.6636	0.5600	0.4844
1.5	0.8769	0.7548	0.6633	0.5904
Increase between $\theta = 0.5$ And $\theta = 1.5$	0.2115 (31.8%)	0.2705 (55.9%)	0.2881 (75.8%)	0.2811 (90.9%)

Table 3-i - Effects of mean-earnings on the optimal leverage ratio

4.4.3.3 Impact on the default barrier

Table 3c shows the impact changes in mean earnings has on the default barrier. As mean-earnings increase, we would expect the default barrier to decrease for the same current earnings. This rationale can be drawn from the intuitive argument, that, you would be willing to wait longer for a project that gives 2\$ on average than one that gives 1\$ on average. Consequently, making you willing to take increased losses in the short term.

Nevertheless, there are several factors impacting the default barrier for the shareholder. As previously explored, the increase in mean earnings does increase the optimal coupon, which in line will impact the default barrier upwards. In this case, the increase on the optimal coupon is dominant and dominates the isolated downward impact increased mean earnings has on the default barrier. Therefore, we can observe from *Table 3c*, that in this particular case, the default barrier increases as mean earnings increase, due to the dominant nature of increasing optimal coupons.

Sub-Conclusion: The optimal default barrier is, in this case, an increasing function of mean-earnings due to the dominant nature of the impact from changes in the optimal coupon.

Table 3c

Effects of mean-earnings on the default barrier

θ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.5	0.0447	0.0431	0.0321	0.0321
1	0.0842	0.0856	0.0862	0.0868
1.5	0.1164	0.1214	0.1276	0.1276

Table 3-j - Effects of mean-earnings on the default barrier

4.4.4 Volatility as the determinant

4.4.4.1 Impact on the optimal coupon

As observed in *Table 4a*, we see that the optimal coupon can be seen as a decreasing function of volatility. Consequently, the optimal coupon decreases as the volatility of earnings increase. This is in line with the results already presented, where the leverage ratio and default barriers decrease as volatility increases. Conclusively, we observe that as volatility increase, the preference of financing skews toward equity, rather than debt, due to the increased chance of default.

Sub-Conclusion: The optimal coupon is a decreasing function of the volatility of earnings.

Table 4a

Effects of volatility on the optimal coupon

σ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.2	0.6804	0.6878	0.6878	0.6878
0.4	0.6260	0.6267	0.6270	0.6273
0.6	0.6135	0.6135	0.6135	0.6135

Table 4-k - Effects of volatility on the optimal coupon

4.4.4.2 Impact on the optimal leverage ratio

As observed in *Table 4b*, the optimal leverage ratio is negatively related to the volatility of earnings. Nevertheless, as proposed by Sarkar and Zapatero in 2003, the volatility is increasingly important as k decreases and less important as k increases. Furthermore, as observed from *Table 4a*, the impact on leverage seems to be less impactful as current earnings increase. Differently, empirical studies have found that the earnings volatility coefficient is

generally not significant for the optimal leverage ratio, as presented by Titman and Wessels in 1988. (Titman & Wessels, 1988)

Sub-Conclusion: The optimal leverage ratio is a decreasing function of volatility of current earnings. But, as presented by Titman and Wessels & Sarkar and Zapatero in 1988 and 2003, the volatilities impact on optimal leverage might be insignificant.

Table 4b

Effects of volatility on optimal leverage

σ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.2	0.8755	0.7214	0.6095	0.5272
0.4	0.8139	0.6636	0.5600	0.4844
0.6	0.7974	0.6485	0.5603	0.4721

Table 4-l - Effects of volatility on optimal leverage

4.4.4.3 Impact on the default barrier

As observed in Table 4c, and with a similar result as for the optimal leverage ratio, the default barrier decreases as the volatility of earnings increase. The increased risk of the project with increased volatility makes the equity holder more contingent to decrease the default barrier. As expected, increased current earnings does not affect the default barrier with the application of mean reversion. (i.e. $k > 0$).

Sub-Conclusion: The default barrier is a decreasing function of the volatility of earnings.

Table 4c

Effects of volatility on the default barrier

σ	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$
0.2	0.2110	0.2264	0.2264	0.2264
0.4	0.0842	0.0856	0.0862	0.0868
0.6	0.0538	0.0538	0.0538	0.0538

Table 4-m - Effects of volatility on the default barrier

5. Conclusion

The goal of this dissertation was to analyze the trade-off theory with mean-reverting earnings and to observe if this version of the trade-off model is in line with empirical evidence. Empirical evidence has shown a negative relation between earnings and leverage, indicating that as earnings increase the leverage ratio decreases. Traditionally, the traditional trade-off theory of capital structure is seen as predicting the opposite. As result, this theory has been looked as less adequate than others. In particular, the pecking order theory has been seen as superior due to its negative and empirically supported relation between leverage and earnings. This dissertation aims to see if the inclusion of mean-reversion, as discussed by Sarkar and Zapatero in 2001, leads to a different conclusion.

In this dissertation, I reformulate the traditional trade-off theory model to include mean reverting earnings and calculate the optimum leverage ratio under this setting. The original input parameters presented by Sarkar and Zapatero were first considered. The model was solved numerically for the optimal default barrier, optimal coupon level, and finally for the optimal leverage ratio. Furthermore, different initial earnings levels, different speeds of mean-reversion and different mean earnings-levels were considered to observe how the model handles different input values.

This dissertation shows that the introduction of mean-reversion strongly affects the results. Under this setting, the relation between earnings and leverage turns to be in line with the empirical evidence of a negative relationship. This result is contrary to the traditional trade-off theory conclusion but is in line with empirical evidence. This result may give more confidence to those that argue in favor of the trade-off theory of capital structure. Furthermore, I show that the speed of mean reversion carries a significant weight in the estimation of the optimal coupon and thus the optimal leverage ratio. In particular, a higher speed of mean-reversion decreases risk and increases the optimal leverage ratio and optimal coupon.

Nonetheless, there are some limitations that must be mentioned regarding this dissertation. Due to the nature of the numerical analysis, the exact nature of all estimations may contain small discrepancies, which may sway some decimals, but which will not change the overall results. Furthermore, the set values for several parameters while solving for the model provides hardship when estimating the collective consequences of larger changes in several parameters. Furthermore, debt is considered perpetual, which may not be the case in all cases.

In conclusion, researcher should adjust the generalized idea that the trade-off theory is not consistent with data in regard to the earnings-leverage relationship. For practitioners, Sarkar and Zapatero mean-reversion model offers a stable approach to the determination of the optimum leverage ratio.

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Appendixes

Python model code

Importing

```
import os

import glob

import pandas as pd

import matplotlib as mplt

import numpy as np

import io

import openpyxl as xl;

from sympy import symbols, Eq, solve, nsolve

import cmath

from mpmath import *

import math
```

The optimal default boundary and other

```
r=0.07

alpha=0.5

t=0.15

k=0.1

theta=1

sigma=0.4
```

```
C=1
```

```
# ### Setting values for V's
```

```
# In[ ]:
```

```
V_1 = (1-t)/(r+k)
```

```
print (V_1)
```

```
# In[ ]:
```

```
V_2 = ((1-t)*k*theta)/(r*(r+k))
```

```
print(V_2)
```

```
# ### Setting values for gamma
```

```
# In[ ]:
```

```
# Solve the quadratic equation  $ax^{**2} + bx + c = 0$ 
```

```
# import complex math module
```

```
a = (1/2)*(sigma**2)
```

$$b = -(1/2)*(\text{sigma}^{**2})-k$$

$$c = -r$$

calculate the discriminant

$$d = (b^{**2}) - (4*a*c)$$

find two solutions

$$\text{gamma_2} = (-b-\text{cmath.sqrt}(d))/(2*a)$$

$$\text{gamma_1} = (-b+\text{cmath.sqrt}(d))/(2*a)$$

Solving for the optimal default boundary

In[]:

substituting x_L for x in this case, and substituting x_L for y in the hypergeometric function to match

$$x = \text{symbols}("x")$$

$$i = 0$$

for y in $\text{np.linspace}(0.4, 0.5, \text{num}=2000)$:

$$\begin{aligned} \text{expr} = & (\text{hyp1fl}(1-\text{gamma_2}, \quad \quad \quad 3-2*\text{gamma_2} \quad \quad \quad + \\ & (2*k/\text{sigma}^{**2}), (2*k*\text{theta})/((\text{sigma}^{**2})*y))) / (\text{hyp1fl}(-\text{gamma_2}, 2-2*\text{gamma_2} \quad \quad \quad + \\ & (2*k/\text{sigma}^{**2}), (2*k*\text{theta})/((\text{sigma}^{**2})*y))) + (x*\text{gamma_2} - ((V_1*(x^{**2})/(V_1*x + V_2 - \\ & (1-t)*(C/r)))) / ((\text{gamma_2}*k*\text{theta})/((1-\text{gamma_2})*(\text{sigma}^{**2})+k)) \end{aligned}$$

$$\text{num} = \text{solve}(\text{expr})$$

```

num1 = num[1]

if math.isclose(y, num1, abs_tol = 0.000009):

    if num1>i:

        i=num1

        x_L = i

next

print(f'The optimal default boundary is {x_L:.4f}')

# In [ ]:

x, y = symbols('x,y')

eq1 = Eq((x-y),0)

eq2 = Eq((hyp1fl(1-gamma_2, 3-2*gamma_2 +
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y)))/(hyp1fl(-gamma_2,2-2*gamma_2 +
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y))) + (x*gamma_2-((V_1*(x**2))/(V_1*x+V_2-
(1-t)*(C/r)))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k)),0)

# In [ ]:

sol_dict = solve((eq1,eq2), (x, j))

print(sol_dict)

# ## Debt capacity

```

```
# In[ ]:
```

```
x_0=1
```

```
x=symbols("x")
```

```
k=0.1
```

```
# Setting values for V_
```

```
V_1 = (1-t)/(r+k)
```

```
V_2 = ((1-t)*k*theta)/(r*(r+k))
```

```
# Solve the quadratic equation  $ax^{**2} + bx + c = 0$ 
```

```
# import complex math module
```

```
a = (1/2)*(sigma**2)
```

```
b = -(1/2)*(sigma**2)-k
```

```
c = -r
```

```
# calculate the discriminant
```

```
d = (b**2) - (4*a*c)
```

```
# find two solutions
```

```
gamma_2 = (-b-cmath.sqrt(d))/(2*a)
```

```
gamma_1 = (-b+cmath.sqrt(d))/(2*a)
```

```
# Setting dummies
```

```
i=0
```

```
p=0
```

```
q=0
```

```
j=0
```

```
eq=0
```

```
# Maximizing debt capacity
```

```
for C in np.linspace(0.60,1, num=100):
```

```
    eq1 = Eq((x-y),0)
```

```
    eq2 = Eq((hyp1fl(1-gamma_2, 3-2*gamma_2 +  
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y)))/(hyp1fl(-gamma_2,2-2*gamma_2 +  
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y))) + (x*gamma_2-((V_1*(x**2))/(V_1*x+V_2-  
(1-t)*(C/r)))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k)),0)
```

```
    sol_dict = solve((eq1,eq2), (x, y))
```

```
    sol_dict=sol_dict[1]
```

```

x_L=sol_dict[1]

m1=hyp1f1(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*x_0))

m2=hyp1f1(-gamma_2,2-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*x_L))

debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

FV=((t*C)/r)+V_2+(V_1*x_0)-
(((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

if debtvalue > i:

    i=debtvalue

    p=C

    q=x_L

    j=FV

    eq=j-i

next

print(f'Maximum debt value is {i:.4f} with a Coupon of {p:.4f}, a default boundary of {q:.4f},
a firm value of {j:.4f} and equity of {eq:.4f} ')

```

```
# In[ ]:
```

```
x_0=1
```

```
k=0.1
```

```
V_1 = (1-t)/(r+k)
```

$$V_2 = ((1-t)*k*theta)/(r*(r+k))$$

Setting dummies

i=0

p=0

q=0

j=0

eq=0

for C in np.linspace(0.60,1,num=40):

 x=symbols("x")

 for y in np.linspace(0.0001, 0.6000, num=50):

 expr=(hyp1fl(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y)))/(hyp1fl(-gamma_2, 2-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y))) + (x*gamma_2 - ((V_1*(x**2))/(V_1*x + V_2 - (1-t)*(C/r))))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k))

 m1=hyp1fl(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*1))

 m2=hyp1fl(-gamma_2, 2-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y))

 num = solve(expr)

 num1 = num[1]

 if math.isclose(y, num1, abs_tol = 0.009):

 x_L = num1

 debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

```
FV=((t*C)/r)+V_2+(V_1*x_0)-  
(((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))
```

```
print(debtvalue)
```

```
print(C)
```

```
print(y)
```

```
print(FV)
```

```
if debtvalue > i:
```

```
    i=debtvalue
```

```
    p=C
```

```
    q=x_L
```

```
    j=FV
```

```
    eq=j-i
```

```
next
```

```
next
```

```
# Exporting to Excel and producing the graph from there
```

```
# In[ ]:
```

```
print(f'Maximum debt value is {i:.4f} with a Coupon of {p:.4f}, a default boundary of {q:.4f},  
a firm value of {j:.4f} and equity value of {eq:.4f}')
```

```
# In[ ]:
```

```
# Finding unlevered firm value
```

```
x_0=1
```

```
V_1 = (1-t)/(r+k)
```

```
V_2 = ((1-t)*k*theta)/(r*(r+k))
```

```
V_unlevered = V_1*x_0+V_2
```

```
print(f'The unlevered firm value is {V_unlevered:.4f}')
```

```
# In[ ]:
```

```
# Setting k to 0.2
```

```
x_0=1
```

```
k=0.2
```

```
V_1 = (1-t)/(r+k)
```

```
V_2 = ((1-t)*k*theta)/(r*(r+k))
```

```

a = (1/2)*(sigma**2)
b = -(1/2)*(sigma**2)-k
c = -r

# calculate the discriminant
d = (b**2) - (4*a*c)

# find two solutions
gamma_2 = (-b-cmath.sqrt(d))/(2*a)
gamma_1 = (-b+cmath.sqrt(d))/(2*a)

for C in np.linspace(0.75,1,num=40):
    x=symbols("x")

    for y in np.linspace(0.0001, 0.6000, num=50):
        expr=(hyp1f1(1-gamma_2,
                    3-2*gamma_2
                    (2*k/sigma**2),(2*k*theta)/((sigma**2)*y)))/(hyp1f1(-gamma_2,2-2*gamma_2
                    (2*k/sigma**2),(2*k*theta)/((sigma**2)*y))) + (x*gamma_2-((V_1*(x**2))/(V_1*x+V_2-
                    (1-t)*(C/r)))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k))

        m1=hyp1f1(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*1))
        m2=hyp1f1(-gamma_2,2-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*y))

        num = solve(expr)

        num1 = num[1]

        if math.isclose(y, num1, abs_tol = 0.009):
            x_L = num1

```

```

debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

FV=((t*C)/r)+V_2+(V_1*x_0)-
(((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

print(debtvalue)

print(C)

print(y)

print(FV)

if debtvalue > i:

    i=debtvalue

    p=C

    q=x_L

    j=FV

    eq=j-i

next

next

# Exporting to Excel and producing the graph from there

# In [ ]:

print(f'Maximum debt value is {i:.4f} with a Coupon of {p:.4f}, a default boundary of {q:.4f},
a firm value of {j:.4f} and equity value of {eq:.4f} ')

```

```
# ## Firm Value
```

```
# In[ ]:
```

```
x_0=1
```

```
k=0.1
```

```
V_1 = (1-t)/(r+k)
```

```
V_2 = ((1-t)*k*theta)/(r*(r+k))
```

```
for C in np.linspace(0.60,0.64,num=80):
```

```
    x=symbols("x")
```

```
    for y in np.linspace(0.03, 0.14, num=250):
```

```
        expr=(hyp1fl(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y)))/(hyp1fl(-gamma_2, 2-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y))) + (x*gamma_2 - ((V_1*(x**2))/(V_1*x + V_2 - (1-t)*(C/r))))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k))
```

```
        m1=hyp1fl(1-gamma_2, 3-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*1))
```

```
        m2=hyp1fl(-gamma_2, 2-2*gamma_2 + (2*k/sigma**2), (2*k*theta)/((sigma**2)*y))
```

```
        num = solve(expr)
```

```

num1 = num[1]

if math.isclose(y, num1, abs_tol = 0.00009):

    x_L = num1

    debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

    FV=((t*C)/r)+V_2+(V_1*x_0)-
    (((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

    print(debtvalue)

    print(C)

    print(y)

    print(FV)

next

next

# Plotting these values into excel and plotting graph shown in Figure 2

# In[ ]:

# Moving in to find the optimal value

x_0=1

k=0.1

```

$$V_1 = (1-t)/(r+k)$$

$$V_2 = ((1-t)*k*\theta)/(r*(r+k))$$

$$i=0$$

$$j=0$$

$$p=0$$

$$q=0$$

$$eq=0$$

for C in np.linspace(0.62,0.63,num=100):

$$x=symbols("x")$$

for y in np.linspace(0.08, 0.085, num=50):

$$\begin{aligned} \text{expr} = & \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*\theta)/((\text{sigma}^{**2})*y)) / (\text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2, \\ & (2*k/\text{sigma}^{**2}), (2*k*\theta)/((\text{sigma}^{**2})*y))) + (x*\text{gamma}_2 - ((V_1*(x^{**2}))/((V_1*x + V_2 - \\ & (1-t)*(C/r)))) / ((\text{gamma}_2*k*\theta)/((1-\text{gamma}_2)*(\text{sigma}^{**2})+k)) \end{aligned}$$

$$m1 = \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2 + (2*k/\text{sigma}^{**2}), (2*k*\theta)/((\text{sigma}^{**2})*1))$$

$$m2 = \text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2 + (2*k/\text{sigma}^{**2}), (2*k*\theta)/((\text{sigma}^{**2})*y))$$

$$\text{num} = \text{solve}(\text{expr})$$

$$\text{num1} = \text{num}[1]$$

if math.isclose(y, num1, abs_tol = 0.00009):

$$x_L = \text{num1}$$

```

    debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

    FV=((t*C)/r)+V_2+(V_1*x_0)-
    (((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

    if FV > i :

        p=debtvalue

        q=x_L

        j=C

        i=FV

        eq=i-p

    next

next

print(f'Maximum firm value is {i:.4f} with a debtvalue of {p:.4f}, an equity value of {eq:.4f},
a default boundary of {q:.4f} and a coupon of {j:.4f} ')

```

Optimal leverage ratio

```

### Changes in earnings and its impact on optimal C

```

```

# In[3]:

```

```

import os

import glob

import pandas as pd

```

```
import matplotlib as mpl

import numpy as np

import io

import openpyxl as xl;

from sympy import symbols, Eq, solve, nsolve, Symbol

import cmath

from mpmath import *

import math

from scipy.optimize import fsolve

r=0.07

alpha=0.5

t=0.15

theta=1

sigma=0.4

C=1

# ## Testing for k = 0.1

# In[ ]:

k=0.1

a = (1/2)*(sigma**2)
```

$$b = -(1/2)*(\sigma^{**2})-k$$

$$c = -r$$

$$d = (b^{**2}) - (4*a*c)$$

$$\text{gamma_2} = (-b-\text{cmath.sqrt}(d))/(2*a)$$

$$\text{gamma_1} = (-b+\text{cmath.sqrt}(d))/(2*a)$$

$$V_1 = (1-t)/(r+k)$$

$$V_2 = ((1-t)*k*\text{theta})/(r*(r+k))$$

x=0.5

In[30]:

x_0=0.5

k=0.1

$$V_1 = (1-t)/(r+k)$$

$$V_2 = ((1-t)*k*\text{theta})/(r*(r+k))$$

i=0

j=0

p=0

q=0

eq=0

for C in np.linspace(0.62,0.63,num=100):

 x=symbols("x")

 for y in np.linspace(0.084, 0.087, num=31):

 expr=(hyp1f1(1-gamma_2, 3-2*gamma_2 +
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y)))/(hyp1f1(-gamma_2,2-2*gamma_2 +
(2*k/sigma**2),(2*k*theta)/((sigma**2)*y))) + (x*gamma_2-((V_1*(x**2))/(V_1*x+V_2-
(1-t)*(C/r))))/(gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k))

 m1=hyp1f1(1-gamma_2, 3-2*gamma_2 +
(2*k/sigma**2),(2*k*theta)/((sigma**2)*x_0))

 m2=hyp1f1(-gamma_2,2-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*y))

 num = solve(expr)

 num1 = num[1]

 if math.isclose(y, num1, abs_tol = 0.00009):

 x_L = num1

 debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

 FV=((t*C)/r)+V_2+(V_1*x_0)-
 (((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

 if FV > i :

```

        p=debtvalue

        q=x_L

        j=C

        i=FV

        eq=i-p

    next

next

print(f'Maximum firm value is {i:.4f} with a debtvalue of {p:.4f}, an equity value of {eq:.4f},
a default boundary of {q:.4f} and a coupon of {j:.4f} ')

```

Determinants of Optimal C

```

import os

import glob

import pandas as pd

import matplotlib as mplt

import numpy as np

import io

import openpyxl as xl;

from sympy import symbols, Eq, solve, nsolve, Symbol

import cmath

from mpmath import *

import math

from scipy.optimize import fsolve

r=0.07

```

alpha=0.5

t=0.15

theta=1

sigma=0.4

C=0.6260

k=0.1

a = (1/2)*(sigma**2)

b = -(1/2)*(sigma**2)-k

c = -r

d = (b**2) - (4*a*c)

gamma_2 = (-b-cmath.sqrt(d))/(2*a)

gamma_1 = (-b+cmath.sqrt(d))/(2*a)

V_1 = (1-t)/(r+k)

V_2 = ((1-t)*k*theta)/(r*(r+k))

x=1

x_0=1

k=0.1

FV_1=13.4772

$$V_1 = (1-t)/(r+k)$$

$$V_2 = ((1-t)*k*theta)/(r*(r+k))$$

i=0

j=0

p=0

q=0

eq=0

for C in np.linspace(0.6260,0.6260,num=1):

x=symbols("x")

for y in np.linspace(0.084, 0.087, num=31):

$$\begin{aligned} \text{expr} = & \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y)) / (\text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y))) + \\ & (x*\text{gamma}_2 - ((V_1*(x^{**2}))/((V_1*x + V_2 - (1-t)*(C/r)))) / ((\text{gamma}_2*k*theta)/((1-\text{gamma}_2)*(\text{sigma}^{**2})+k)) \end{aligned}$$

$$\text{m1} = \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*x_0))$$

$$\text{m2} = \text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2 + (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y))$$

num = solve(expr)

num1 = num[1]

if math.isclose(y, num1, abs_tol = 0.00009):

```

x_L = num1

debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

FV=((t*C)/r)+V_2+(V_1*x_0)-
(((t*C)/r)+alpha*(V_1*x_L+V_2))*(m1/m2)*((1/x_L)**gamma_2)

if FV > i :

    p=debtvalue

    q=x_L

    j=C

    i=FV

    eq=i-p

    loss_percentage=((i-FV_1)/FV_1)*100

next

next

print(f'Maximum firm value is {i:.4f} with a debtvalue of {p:.4f}, an equity value of {eq:.4f},
a default boundary of {q:.4f} and a coupon of {j:.4f} ')

print(loss_percentage)

```

Testing mean-earnings as the determinant

```
# ## Testing for theta = 0.5
```

```
import os
```

```
import glob
```

```
import pandas as pd
```

```
import matplotlib as mplt
```

```
import numpy as np
```

```

import io

import openpyxl as xl;

from sympy import symbols, Eq, solve, nsolve, Symbol

import cmath

from mpmath import *

import math

from scipy.optimize import fsolve

r=0.07

alpha=0.5

t=0.15

theta=0.5

sigma=0.4

k=0.1

a = (1/2)*(sigma**2)

b = -(1/2)*(sigma**2)-k

c = -r

d = (b**2) - (4*a*c)

gamma_2 = (-b-cmath.sqrt(d))/(2*a)

gamma_1 = (-b+cmath.sqrt(d))/(2*a)

V_1 = (1-t)/(r+k)

V_2 = ((1-t)*k*theta)/(r*(r+k))

```

```

#### x=0.5

x_0=0.5

k=0.1

V_1 = (1-t)/(r+k)

V_2 = ((1-t)*k*theta)/(r*(r+k))

i=0

j=0

p=0

q=0

eq=0

for C in np.linspace(0.28,0.34,num=50):

    x=symbols("x")

    for y in np.linspace(0.01, 0.1, num=50):

        expr=(hyp1f1(1-gamma_2,
                    3-2*gamma_2
                    (2*k/sigma**2),(2*k*theta)/((sigma**2)*y)))/(hyp1f1(-gamma_2,2-2*gamma_2
                    (2*k/sigma**2),(2*k*theta)/((sigma**2)*y))) + (x*gamma_2-((V_1*(x**2))/(V_1*x+V_2-
                    (1-t)*(C/r)))/((gamma_2*k*theta)/((1-gamma_2)*(sigma**2)+k))

        m1=hyp1f1(1-gamma_2,
                    3-2*gamma_2
                    (2*k/sigma**2),(2*k*theta)/((sigma**2)*x_0))

        m2=hyp1f1(-gamma_2,2-2*gamma_2 + (2*k/sigma**2),(2*k*theta)/((sigma**2)*y))

        num = solve(expr)

        num1 = num[1]

        if math.isclose(y, num1, abs_tol = 0.00009):

            x_L = num1

```

```

    debtvalue=(C/r)+((1-alpha)*(V_1*x_L+V_2)-(C/r))*(m1/m2)*((1/x_L)**gamma_2)

    FV=((t*C)/r)+V_2+(V_1*x_0)-
    (((t*C)/r)+alpha*(V_1*x_L+V_2))*((m1/m2)*((1/x_L)**gamma_2))

    if FV > i :

        p=debtvalue

        q=x_L

        j=C

        i=FV

        eq=i-p

    next

next

print(f'Maximum firm value is {i:.4f} with a debtvalue of {p:.4f}, an equity value of {eq:.4f},
a default boundary of {q:.4f} and a coupon of {j:.4f} ')

```

Testing Volatility as the determinant

```

### Testing for sigma = 0.2

import os

import glob

import pandas as pd

import matplotlib as mpl

import numpy as np

import io

import openpyxl as xl;

from sympy import symbols, Eq, solve, nsolve, Symbol

```

```

import cmath

from mpmath import *

import math

from scipy.optimize import fsolve

r=0.07

alpha=0.5

t=0.15

theta=1

sigma=0.2

k=0.1

a = (1/2)*(sigma**2)

b = -(1/2)*(sigma**2)-k

c = -r

d = (b**2) - (4*a*c)

gamma_2 = (-b-cmath.sqrt(d))/(2*a)

gamma_1 = (-b+cmath.sqrt(d))/(2*a)

V_1 = (1-t)/(r+k)

V_2 = ((1-t)*k*theta)/(r*(r+k))

### x=0.5

# In[24]:

x_0=0.5

k=0.1

```

$$V_1 = (1-t)/(r+k)$$

$$V_2 = ((1-t)*k*theta)/(r*(r+k))$$

$$i=0$$

$$j=0$$

$$p=0$$

$$q=0$$

$$eq=0$$

for C in np.linspace(0.66,0.70,num=50):

$$x=symbols("x")$$

for y in np.linspace(0.20, 0.23, num=50):

$$\begin{aligned} \text{expr} = & \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y)) / (\text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y))) + \\ & (x*\text{gamma}_2 - ((V_1*(x^{**2}))/((V_1*x + V_2 - (1-t)*(C/r)))) / ((\text{gamma}_2*k*theta)/((1-\text{gamma}_2)*(\text{sigma}^{**2})+k)) \end{aligned}$$

$$m1 = \text{hyp1fl}(1-\text{gamma}_2, 3-2*\text{gamma}_2, (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*x_0))$$

$$m2 = \text{hyp1fl}(-\text{gamma}_2, 2-2*\text{gamma}_2 + (2*k/\text{sigma}^{**2}), (2*k*theta)/((\text{sigma}^{**2})*y))$$

$$\text{num} = \text{solve}(\text{expr})$$

$$\text{num1} = \text{num}[1]$$

if math.isclose(y, num1, abs_tol = 0.00009):

$$x_L = \text{num1}$$

$$\text{debtvalue} = (C/r) + ((1-\text{alpha})*(V_1*x_L + V_2) - (C/r)) * (m1/m2) * ((1/x_L)**\text{gamma}_2)$$

$$\begin{aligned} \text{FV} = & ((t*C)/r) + V_2 + (V_1*x_0) - \\ & (((t*C)/r) + \text{alpha}*(V_1*x_L + V_2)) * ((m1/m2) * ((1/x_L)**\text{gamma}_2)) \end{aligned}$$

if FV > i :

```
p=debtvalue
```

```
q=x_L
```

```
j=C
```

```
i=FV
```

```
eq=i-p
```

```
next
```

```
next
```

```
print(f'Maximum firm value is {i:.4f} with a debtvalue of {p:.4f}, an equity value of {eq:.4f},  
a default boundary of {q:.4f} and a coupon of {j:.4f} ')
```