Portfolio Optimization Under 'at-risk' Constraints

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Abstract

The financial crisis of 2008 brought with it a great interest in downside risk. This dissertation focus on portfolio optimization under downside risk constraints. We maximize the expected return subject to the level of risk, which is defined as Value at Risk (VaR) or Conditional Value at Risk (CVaR) above the risk free rate on the initial wealth. Since this model does not depend on distributional assumptions for the returns, we are able to evade the shortcomings of overestimation or underestimation of risk. In an out-of-sample exercise between 1994 and 2014, we show that our VaR and CVaR strategies yield an annualized Sharpe ratio of 0.67 and 0.63, respectively, which compares well to the S&P500 that yields an annualized Sharpe ratio of 0.47. Additionally, we find evidence that our downside risk model for portfolio optimization exhibits better results during recessions when comparing its performance with several benchmarks. This implies that our model can be viewed as a risk mitigation strategy.

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1. Introduction

The relation between the return of an investment and its underlying risk has long been an issue of great interest in finance. The incorporation of this concept in portfolio optimization has its origins in the mean-variance framework presented by the Nobel laureate, Harry Markowitz. The simplicity and intuition presented in Markowitz (1952) have led to the foundations of one of the greatest contributions to financial theory, the Modern Portfolio Theory (MPT). In this model the optimal asset allocation is determined by maximizing the expected return subject to a certain level of risk (or minimizing the risk for a given expected return), where risk is defined in terms of the possible variation of expected portfolio returns, i.e., standard deviation. Despite its popularity and widespread use, the mean-variance criterion has also received substantial criticism due to its main assumptions. Since asymmetrical behavior is preferred and returns are often non-normally distributed, this model is likely do give rise to inefficient portfolios and it would be desirable to use a downside risk measure instead. Note that both Markowitz and William Sharpe, the other originator of MPT, acknowledge these limitations from the very beginning and suggested another measure of risk, which focus on the downside.

"Under certain conditions, the mean-variance approach can be shown to lead to unsatisfactory predictions of behavior. Markowitz suggests that a model based on the semivariance would be preferable; in light of the formidable computational problems, however, he bases his analysis on the variance and standard deviation." – Sharpe (1964)

This dissertation focus on portfolio optimization under downside risk constraints, following the methodology of Campbell et al. (2001). They develop a market equilibrium model for asset allocation under downside risk constraints that does not require any distributional assumption. We define two alternative downside risk constraints and test this framework under different market conditions. We find that this strategy performs well under recession periods when compared to the benchmarks defined, meaning that it is a way of mitigating risk.

The global financial crisis of 2008, which was originated by the subprime mortgage crisis, brought with it a great interest in downside risk within the financial services industry and the academic world. However, over the past decades academic researchers have already been giving relevance to the idea of considering downside risk constraints for asset allocation, with Value at Risk (VaR) being one of the most considered measures (Hull, 2006). For example, the Basel Committee on Banking Supervision recommends

VaR to measure market risk and demands U.S. banks to determine the minimal capital requirements using this tool. Furthermore, VaR is widely used by fund managers, dealers, corporate treasurers and financial institutions. The popularity of this measure is explained by the fact that it is easily understood. It is defined as the maximum expected loss of a portfolio over a given horizon for a given confidence level.[1] For example, a 95% VaR for a 1-month holding period implies that the maximum loss incurred over the next month should not exceed the VaR limit more than five times in every 100 cases. However, researchers have been criticizing the use of VaR as a measure of risk due to its shortcomings. Beder (1995) stressed out that VaR is extremely sensitive to parameter choice and Artzner et al. (1999) classify VaR as not being a "coherent" measure of risk because it fails to satisfy the "subadditivity property". This means that the VaR of a portfolio with two securities may exceed the sum of each of the securities' VaR individually. Given these drawbacks, different downside risk measures were proposed as alternatives. Li (1999) suggested a new approach to estimate VaR that also includes skewness and kurtosis. Favre and Galeano (2002) developed the Modified Value at Risk, a new estimation method for VaR in which they use a Corner-Fisher expansion. Rockafellar and Uryasev (2000) proposed the use of Conditional Value at Risk (CVaR), which is defined as the expected value of losses exceeding VaR over a given horizon for a given confidence level. The CVaR is classified as a "coherent" risk measure and several studies state its best performance when compared to the VaR. For example, Alexander and Baptista (2004) compare the performance of VaR and CVaR as constraints and demonstrate that CVaR is more effective on the mean-variance model, especially when a risk-free security is considered.

A brief description of recent literature is given in what follows. Ciliberti, Kondor and Mézart (2007) address portfolio optimization problem under the CVaR constraint using a linear programming setting. They show that when the time series period is too short, the problem is ill posed since it results in unbounded solutions. In a derivatives framework, Alexander, Coleman and Li (2006) also conclude that optimal portfolio choice under VaR and CVaR constraints is typically ill-posed. Therefore, they propose to include a proportional cost in the minimization problem. When comparing a computationally method with the traditional linear programming approach, the conclusion is that the first one leads to a more efficient solution for large scale optimization problems. Adam,

[1] Generally, the confidence levels range from 95% to 99%. The Basel Committee recommends the 99% VaR for a 10-day holding period to establish a bank's capital adequacy requirements.

Houkari and Laurent (2008) use hedge funds data due to their non-Gaussian properties to compare the solutions of portfolio optimization under different risk constraints. They show that risk measures which emphasize large losses result in more diversified portfolios. Alexander, Baptista and Yan (2007) study the influence of adding a VaR or a CVaR constraint to the mean-variance model. The results show that both VaR and CVaR-constrained portfolios are mean-variance efficient. Furthermore, the VaR constraint is less effective to restrict large losses in this framework, which supports the use of CVaR. Brandtner (2012) analyze portfolio optimization under CVaR and spectral risk measures constraints, comparing it with the standard mean-variance approach. A spectral risk measure is defined as “a weighted average of the quantiles of the distribution of the returns using a non-increasing weight function called the spectrum”. They conclude that spectral risk measures lead to corner solutions and when considering a risk free asset, the diversification is never optimal. Abad and Iyenger (2012) also focus on spectral risk constraints, proposing an efficient iterative algorithm for portfolio optimization.

We contribute to the literature on portfolio optimization under shortfall constraints by studying the model for asset allocation under downside risk constraints developed by Campbell et al. (2001). They define risk in terms of the VaR above the risk free rate on the initial wealth and select the portfolio by maximizing an index similar to the Sharpe ratio, i.e., by maximizing the expected return subject to the level of risk. Instead of relying only on VaR as the downside risk measure, we also consider CVaR, a “coherent” risk measure that is shown in literature to achieve better results than VaR. We test this framework by comparing its performance with several benchmarks under different market conditions: different industries and different economic periods. The results show that the downside risk constraint model yields a better performance during recessions, meaning that it is a way of mitigating risk. The different industry typed do not have impact on the performance of our strategy. We also test the impact of non-normal returns and conclude that assuming a distribution for the returns that differs from the historical one will result in underestimation or overestimation of risk.

The rest of this study is as follows. In Section 2 we introduce the methodology of portfolio optimization. Section 3 we present the data used. In Section 4 we show the impact of non-normal returns in the asset allocation exercise and in Section 5 we analyze the performance of our strategy over time. Section 6 concludes.

2. The Portfolio Selection Model

In this section, we briefly explain the model for asset allocation under downside risk constraints based on the methodology of Arzac and Bawa (1977) and Campbell et al. (2001).[2] We expand the set of downside risk measures considered by using VaR and CVaR measures. Taking into consideration that the logic is the same for both risk measures, as a simplification we present the model considering only the VaR measure.

The portfolio selection model is subject to a limited loss, which is set by the investor over a specified period. In other words, the optimal portfolio is derived such that the expected maximum loss does not exceed the downside risk constraint for a given horizon and confidence level. This threshold could be set by the private investor according to her degree of risk aversion or by the risk management department, in order for the financial institution to comply with capital requirements, such as Basel. It is important to highlight that this methodology avoids the limitations of expected utility theory regarding the choice of the degree of risk aversion that an investor exhibits. Notwithstanding, we take into account the degree of risk aversion through the confidence level that defines the threshold.

2.1. The Portfolio Problem

In order to define the portfolio problem we consider the choice among several risky assets and a secure asset. Suppose that an investor has an amount W_0 to invest over a horizon T and that she is able to lend or borrow any amount B at the risk free rate r_f. Note that $B < 0$ represents lending and $B > 0$ represents borrowing. There are n available assets and w_i is defined as the fraction invested in the risky asset i.[3] Let $P_{i,t}$ denote the price of asset i at time t. Being the current decision period at $t = 0$, the budget constraint is given by:

$$W_0 + B = \sum_{i=1}^{n} w_i P_{i,0} \tag{1}$$

The investor needs to choose the weights of each asset such that the expected maximum loss does not exceed the VaR target. Therefore, the downside risk constraint can be defined as follows:

$$Pr\{W_0 - W_{T,p} \geq VaR^*\} \leq (1 - \alpha), \tag{2}$$

[2] See Arzac and Bawa (1977) and Campbell et al. (2001) for a more detailed explanation of the model.
[3] Note that the sum of all the fractions w_i must be equal to one.

where Pr is the probability conditional on the information set available at time zero for portfolio p, VaR* denotes the desired VaR level and α represents the confidence level. Note that, the investor's level of risk aversion is reflected in the desired VaR and the confidence level associated to it.

The investor aims to maximize the wealth at the end of the investment horizon. Defining r_p as the expected total return of portfolio p in period T, we can express the expected final wealth from investing in portfolio p as:

$$E_0(W_{T,p}) = (W_0 + B)(1 + r_p) - B(1 + r_f) \tag{3}$$

Therefore, the portfolio allocation problem is defined as the maximization of the expected final wealth by choosing the amount invested in the risky assets and in the risk free asset under the constraints (1) and (2) defined above.

2.2 The Optimal Portfolio

According to Arzac and Bawa (1977), this portfolio problem can be solved in two different stages. First, the investor faces the choice of the optimal fractions to invest in the risky assets, which are independent of wealth and borrowing/lending. Second, the investor needs to choose the allocation between the risky portfolio and the risk free asset. In these sense, the investor faces two different problems.

Let us focus our attention on the first-stage of the problem. By substituting the expected final wealth (Eq. (3)) in the downside risk constraint (Eq. (2)), the following result is obtained:

$$Pr\left\{r_p \leq \frac{B.r_f - VaR^*}{W_0 + B}\right\} \leq (1 - \alpha) \tag{4}$$

Considering that $q_{\alpha,p}$ denotes the α quantile of the expected return distribution for portfolio p, Eq. (4) can be transformed into the following expression:

$$q_{\alpha,p} \leq \frac{B * r_f - VaR^*}{W_0 + B} \tag{5}$$

Taking into account that the investor aims to maximize the expected final wealth, by substituting for $W_0 + B$ from Eq. (5) in Eq. (3) one obtains:

$$E_0(W_{T,p}) = W_0(1 + r_f) + W_0(W_0.r_f - VaR^*)\frac{r_p - r_f}{W_0.r_f - W_0.q_{\alpha,p}} \tag{6}$$

Note that the only variables depending on the weights are r_p and $q_{\alpha,p}$. Therefore, the first stage problem can be redefined as:

$$\max_{w_i} S_p = \frac{r_p - r_f}{W_0 . r_f - W_0 . q_{\alpha,p}} \tag{7}$$

It should be noticed that the choice of the optimal portfolio is independent of wealth. Although initial wealth is in the denominator of expression (7), it is a constant, meaning that the asset allocation problem is not affected. Nonetheless, it is convenient to express the denominator with the initial wealth, since it facilitates the interpretation. S_p can be described as the ratio of the expected risk premium of portfolio p and its risk, which is measured as the maximum expected loss on portfolio p (which is incurred with probability $1 - \alpha$) relative to the risk free rate. Note that the initial wealth multiplied by the negative quantile of the return distribution represents the portfolio VaR for a given confidence level. As a result, it is possible to express the risk taken by the investor as:

$$\varphi_{\alpha,p} = W_0 . r_f - VaR_{\alpha,p} \tag{8}$$

This measure of risk focus on the risk free rate as a benchmark, which is in accordance with investors' behavior. $\varphi_{\alpha,p}$ can be defined as the potential opportunity cost of investing in risky assets, since it measures the potential losses with respect to the risk free rate of return. In this sense, the first stage problem can be written as follows:

$$\max_{w_i} S_p = \frac{r_p - r_f}{\varphi_{\alpha,p}} \tag{9}$$

This ratio is similar to the Sharpe Ratio and, consequently, it allows to assess the portfolios' efficiency. Actually, when the expected portfolio returns are assumed to be normally distributed and the risk free is zero, S_p becomes a multiple of the Sharpe Ratio. In this case, the VaR is defined as a multiple of the standard deviation and both measures lead to the same optimal portfolio. When the risk free rate is positive, the difference in the weights of the optimal portfolios is minimal (considering a small time horizon).

In the second stage problem, the investor needs to choose the allocation between the risky portfolio obtained in (9) and the risk free rate. From equation (5) we can see that the amount to borrow or lend is given by:

$$B = \frac{W_0(VaR^* - VaR_{\alpha,p'})}{\varphi'_{\alpha,p'}} \quad (10)$$

where p' represents the optimal portfolio chosen in the first stage by maximizing equation (9). Note that the risky portfolio is independent of the desired VaR, since it uses the estimated portfolio VaR. The desired VaR reflects the degree of risk aversion of the investor and, in order for this limit to be met, the investor may choose the amount to borrow or lend.

It is possible to conclude that the model is independent of distributional assumptions, since the optimal portfolio is achieved by assuming that the investors wish to maximize the expected return subject to a downside risk constraint.

3. Data

Our portfolio is allocated among two risky assets, stocks and bonds, and one riskless asset, the risk-free rate. We consider the framework of an U.S. investor and use the S&P500 composite index and the 10-year U.S. benchmark government bond index as risky assets. The 3-month U.S. Treasury-Bill rate is set as reference for the risk-free rate. We use DataStream to collect the monthly and daily returns from October 1989 until August 2014, which provides us with 298 monthly and 6,495 daily observations.

We also address size and sector in our empirical results. The size effect is taken into account using the Russell 1000 index, a proxy for large market capitalization stocks (large cap), and Russell 2000 index, a reference for small market capitalization stocks (small cap). We use these data from Bloomberg. To analyze the sector effect, in particular how our strategy would perform for energy, financial and technology industries, we use the S&P500 Energy, the S&P500 Banks and the S&P500 Software indexes from Bloomberg. The purpose of the choice of these specific sectors is the coverage of several levels of systematic risk. The energy sector is considered to have a beta lower than 1, which reflects a sector with high stability that is less affected by market fluctuations, whereas the other two sectors present a beta higher than 1, meaning that these industries are highly linked to business cycles. Additionally, the financial crisis of 2008 was sprouted by the financial sector and by allocating the portfolio to this specific industry we are able to evaluate the performance of our strategy under this scenario.

The descriptive statistics for the returns of each asset are presented in Table 1 for both time frequencies: daily and monthly.

Table 1
Summary Statistics

This table provides the summary statistics for the assets used: US 3-month Treasury Bill rate (Risk-Free), 10-year US benchmark government bond index (Bonds), S&P 500 composite returns index (S&P500), Russell 1000 index (Large Cap), Russell 2000 index (Small Cap), S&P 500 Energy index (Energy), S&P 500 Financials index (Financial) and S&P 500 Software index (Tech.). We present the summary statistics from October 1989 to July 2014, which gives us 6,495 daily observations and 298 monthly observations.

	Risk-Free	Bonds	S&P500	Large Cap	Small Cap	Energy	Financial	Tech.
Daily								
Average (%)	0.01	0.02	0.04	0.03	0.03	0.04	0.02	0.04
St. deviation (%)	0.01	0.45	1.13	1.14	1.33	1.48	2.03	1.86
Minimum (%)	0.00	-2.83	-9.46	-9.56	-12.61	-16.88	-23.62	-10.99
Maximum (%)	0.03	4.05	10.96	11.04	8.86	16.96	22.04	15.38
Skewness	0.02	-0.14	-0.27	-0.28	-0.36	-0.28	0.31	0.07
Excess kurtosis	1.81	6.02	12.19	11.76	9.63	14.80	25.77	7.24
Auto Correlation	1.00	0.02	-0.06	-0.02	0.01	-0.05	-0.02	0.00
Sharpe ratio	-	0.05	0.03	0.03	0.03	0.03	0.01	0.02
Perf. Index VaR	-	0.03	0.02	0.02	0.02	0.02	0.01	0.01
Perf. Index CVaR	-	0.02	0.01	0.01	0.01	0.01	0.00	0.01
Percentile - 5%	0.00	-0.72	-1.72	-1.75	-2.08	-2.19	-2.70	-2.97
Percentile - 3%	0.00	-0.87	-2.18	-2.21	-2.58	-2.71	-3.48	-3.65
Percentile - 1%	0.00	-1.26	-3.13	-3.17	-3.79	-3.93	-5.38	-5.10
Monthly								
Average (%)	0.26	0.52	0.75	0.59	0.62	0.72	0.29	0.95
St. deviation (%)	0.19	2.08	4.27	4.32	5.61	5.23	7.21	7.92
Minimum (%)	0.00	-7.36	-18.39	-19.24	-23.45	-19.86	-45.82	-25.40
Maximum (%)	0.65	9.40	10.83	10.58	15.20	16.79	25.00	27.66
Skewness	0.01	-0.04	-0.80	-0.85	-0.76	-0.33	-1.35	-0.20
Excess kurtosis	1.80	4.32	4.67	4.86	4.55	4.28	9.81	4.11
Auto Correlation	1.00	0.10	0.06	0.08	0.12	-0.05	0.10	-0.09
Sharpe ratio	-	0.25	0.18	0.14	0.11	0.14	0.04	0.12
Perf. Index VaR	-	0.18	0.10	0.07	0.07	0.09	0.02	0.07
Perf. Index CVaR	-	0.12	0.07	0.05	0.05	0.06	0.02	0.05
Percentile - 5%	0.00	-2.67	-7.35	-7.80	-8.57	-8.13	-11.43	-12.94
Percentile - 3%	0.00	-3.67	-8.64	-8.91	-11.48	-11.42	-13.86	-16.32
Percentile - 1%	0.00	-4.81	-11.39	-11.40	-15.59	-13.54	-26.25	-21.02

Unsurprisingly, the government bond index is less volatile when compared to the stock market indexes for both frequencies. During the sample period, its annualized standard deviation is lower than the one of S&P500 for both frequencies, but due to the risk-return tradeoff, the government bond index also reveals a lower average return than the S&P 500 index for both frequencies. Notwithstanding, when taking into account risk-adjusted performance measures, the S&P500 yields a slightly lower Sharpe ratio than the government bonds index.

From Table 1 we can see that the size premium is reflected in our sample, i.e., small cap stocks show a higher return than large cap stocks. However, small cap stocks also yield a higher standard deviation, which illustrates the explanation of Fama and French (1993) for this anomaly: small cap stocks may expose investors to some undiversifiable risk that requires a higher rate of return. Another interesting conclusion is that the financial and technology industries exhibit the highest standard deviations for both frequencies, which is coherent with exhibiting a high beta. Note that the highest risk does not necessarily mean the highest loss, but a higher probability of having a loss. However, as we can see from the table, these industries also seem to be the ones with the most negative percentiles. Focusing on the financial sector, we can see that besides the high risk and the negative percentile, it also exhibits by far the highest excess kurtosis, suggesting that extreme events, especially negative ones, are more likely than a normal distribution suggests. Besides its great exposure to market risk, a rational for these differences might be the fact that this is a highly leveraged sector.

The summary statistics also show the aggregational gaussianity property: as the time scale over which returns are calculated increases, their distribution looks more like a normal distribution. In Table 1 this is highlighted by the decrease in excess kurtosis when we move from daily to monthly returns. Moreover, note that, except for the risk-free rate, the returns time series are not persistent, since the auto correlation is close to zero.

4. Asset Allocation Results

The optimal portfolio that maximizes the performance index in (9) is found by using various combinations of stocks and bonds to estimate the expected return r_p and the downside risk measure for the daily and monthly data frequencies over the sample period. In this section, we allocate our portfolio to the S&P500 composite index, the 10-year U.S. benchmark government bond index and the 3-month U.S. Treasury-Bill rate. In order to evaluate the impact of non-normality, we apply different distributions to the returns: historical, normal and t-student. The desired confidence level has been set at 95% using the historical distribution, so that we can use this benchmark to compare the alternative distributional assumptions considered.[4]

[4] The desired VaR level is the 95% VaR from the empirical distribution and the desired CVaR level is the 95% CVaR from the empirical distribution.

In Table 2 we show the combinations of stocks and bonds for a variety of confidence levels, as well as the associated downside risk measures under the assumption that the returns are distributed as in the past. Note that this is the first step to find the optimal portfolio – to choose the optimal fractions to invest in the risky assets, which is independent of the borrowing/lending decision. Therefore, the choice reflected in Table 2 only takes into account the risky assets allocation and does not incorporate the borrowing/lending decision.

Table 2

Risky Assets' Portfolios Under Empirical Distribution

This table presents the weights and the associated downside risk measure for the asset allocation exercise. The portfolio risk measure is presented as a positive value, since it represents the loss in \$. Optimal portfolios consisting of US stocks and bonds are found by maximizing the risk-return trade-off from equation (9). We use data on the S&P 500 composite return index and the 10-year US benchmark government bond index from DataStream over the period February 1988 - July 2014. The risk-free rate considered is the rate on the last period's three month Treasury bill (0.03%). The VaR and CVaR for \$1000 held in the portfolios are estimated using the historical distribution.

	VaR			CVaR		
Confidence Level (%)	Stocks (%)	Bonds (%)	Risk Measure (\$)[a]	Stocks (%)	Bonds (%)	Risk Measure (\$)[b]
Daily						
95	24.0	76.0	6.0	35.3	64.7	10.1
96	26.6	73.4	6.7	35.1	64.9	10.8
97	24.4	75.6	7.3	33.9	66.1	11.6
98	27.0	73.0	8.8	41.2	58.8	14.6
99	30.1	69.9	11.1	33.6	66.4	15.6
Monthly						
95	27.5	72.5	24.0	28.5	71.5	39.1
96	31.7	68.3	29.0	33.0	67.0	43.1
97	47.1	52.9	38.0	50.7	49.3	58.5
98	39.6	60.4	40.1	47.0	53.0	62.3
99	32.0	68.0	46.3	32.8	67.2	58.1

[a] The portfolio risk measure is given by VaR.

[b] The portfolio risk measure is given by CVaR.

An investor with a VaR target at the 95% confidence level obtains similar results for the daily and monthly frequencies: the optimal allocation is found when 24.0% of his wealth is held in stocks and 76.0% in bonds for the daily time horizon and 27.5% in stocks and 72.5% in bonds for the monthly frequency. The same is not verified when using the CVaR constraint, since the daily results suggest a higher proportion to be invested in stocks than the monthly results (35.3% vs 28.5%, respectively). From Table 2 we can also conclude that a higher holding period results in a higher risk measure, since the monthly risk measures are higher than the daily ones.

Naturally, as the confidence level associated with the risk measures increases, the portfolio risk measure increases as well. A more risk averse investor demands a higher confidence level, which is expected to result in a decrease in the proportion of wealth invested in stocks and an increase in bonds. Nevertheless, due to the use of the historical distribution, the stock proportions are not a monotonic function of the confidence level.

In order to ensure that the desired threshold is met, Equation (10) suggests that the greater the confidence level associated with the risk measure, a higher proportion of the portfolio will be needed to be invested in the riskless asset. This translates the second step of the problem and can be seen as a movement along the Capital Market Line. Table 3 provides the final proportions of the optimal portfolio that maximizes the risk-return trade-off taking into account the decision to borrow or lend.

Table 3
Optimal Portfolios Under Empirical Distribution

This table shows the weights and the associated downside risk measure for the asset allocation exercise assuming that the returns are distributed as in the past. Here, the optimal portfolio's choice includes the decision to borrow or lend, which is given by Equation (10). To estimate the desired VaR and desired CVaR, the historical distribution at the 95% level is used and investors are assumed to hold $1000 in the portfolios. The data used are as described in Table 2.

	VaR Constraint				CVaR Constraint			
Confidence Level (%)	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[a]	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[b]
Daily								
95	24.0	76.0	0.0	6.0	35.3	64.7	0.0	10.1
96	24.1	66.4	9.5	6.0	33.0	60.9	6.1	10.1
97	20.7	64.1	15.2	6.0	30.0	58.5	11.5	10.1
98	20.5	55.5	24.0	6.0	31.5	45.0	23.5	10.1
99	20.7	48.0	31.3	6.0	24.8	49.2	26.0	10.1
Monthly								
95	27.5	72.5	0.0	24.0	28.5	71.5	0.0	39.1
96	27.4	59.0	13.5	24.0	30.3	61.5	8.2	39.1
97	35.0	39.4	25.6	24.0	38.4	37.4	24.2	39.1
98	28.7	43.9	27.4	24.0	34.6	39.1	26.4	39.1
99	22.0	46.7	31.3	24.0	24.9	51.2	23.9	39.1

[a] The portfolio risk measure is given by VaR.

[b] The portfolio risk measure is given by CVaR.

The downside risk constraints at the 95% confidence level imply that the investor will not invest in the riskless asset, since this is the desired threshold. For example, the 95%

daily VaR is $6, which corresponds to the desired VaR defined. In this sense, the VaR constraint is already met and the investor does not require any amount to be borrowed or lent. However, if the investor requires a higher confidence level so that the initial wealth will not drop by more than the threshold, the risk measure associated with the portfolio allocation will be greater than the limit set, resulting in too much risk being taken. As a result, the investor needs to invest in the risk free rate so that less risk is taken and the downside risk constraint is met. From Table 3 we can see that for the confidence levels above 95%, the investor was able to meet the threshold by investing in the risk free asset. Furthermore, the higher the confidence level, the higher the portfolio proportion invested in cash.

Our strategy does not require distributional assumptions, but it is possible to assume a distribution for the returns and calculate the risk measures based on the assumption stablished. Notwithstanding, what would be the advantages or the shortcomings of doing so? Besides the empirical distribution, we consider other two alternatives: the normal distribution, the most common distributional assumption in the Finance literature; and the t-student distribution with 5 degrees of freedom, a fat tailed distribution.[5] The results for both risk measures are shown in Table 4.

We have seen in Table 3 that at the 95% confidence level, the proportion invested in cash was zero. However, in Table 4 this is not verified under the assumption that returns follow other distributions than the historical one, since the desired threshold is given by the 95% risk measure from the empirical distribution. For example, for the daily frequency, the desired CVaR is $10.1 and we can see that at the 95% confidence level the CVaR is $10.4 under the normal distribution and $11.7 for the t-student assumption. This means that the downside risk measure exceeds the threshold and the investor needs to invest in the risk-free asset in order to meet the downside constraint.

[5] The smaller the number of degrees of freedom considered, the fatter the tails of the distribution and the greater the deviation from normality. We follow the findings of Campbell et al. (2001) and consider 5 degrees of freedom. They adopt the approach of Huisman et al. (2001) to correctly estimate the degrees of freedom for the t-student distribution in small samples, through the usage of tail index estimation techniques. The results show that the use of 5 degrees of freedom throughout the empirical analysis provides consistent results.

Table 4

Optimal Portfolios Under Normality and T-student

This table shows the weights for the asset allocation exercise assuming that the expected returns are normally and student-t distributed with 5 degrees of freedom. The optimal portfolio's choice includes the decision to borrow or lend, which is given by Equation (10). To estimate the desired VaR, CVaR and Tail Risk measures, the historical distribution at the 95% level is used and investors are assumed to hold $1000 in the portfolios. The data used are as described in Table 2.

Panel A: VaR Constraint

	Normality				Student-t			
Confidence Level (%)	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[a]	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[a]
Daily								
95	23.7	71.8	4.5	6.0	20.2	61.4	18.4	6.0
96	22.4	68.0	9.6	6.0	19.2	58.4	22.4	6.0
97	21.2	64.3	14.5	6.0	18.2	55.5	26.3	6.0
98	20.0	60.6	19.4	6.0	17.4	52.2	30.5	6.0
99	18.6	56.5	24.9	6.0	16.2	48.7	35.1	6.0
Monthly								
95	30.2	67.9	1.9	24.0	25.3	56.8	17.9	24.0
96	28.4	63.8	7.9	24.0	23.9	53.8	22.2	24.0
97	26.7	59.9	13.5	24.0	22.6	50.9	26.5	24.0
98	25.0	56.1	18.9	24.0	21.3	48.0	30.7	24.0
99	23.1	52.0	24.9	24.0	19.9	44.7	35.5	24.0

Panel B: CVaR Constraint

	Normality				Student-t			
Confidence Level (%)	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[b]	Stocks (%)	Bonds (%)	Cash (%)	Risk Measure ($)[b]
Daily								
95	34.3	63.3	2.4	10.1	26.9	61.4	11.7	10.1
96	32.5	60.3	7.1	10.1	26.0	58.8	15.3	10.1
97	30.9	57.5	11.6	10.1	24.9	54.5	20.7	10.1
98	26.7	60.9	12.4	10.1	21.6	55.5	22.9	10.1
99	26.0	55.0	19.0	10.1	20.0	50.6	29.5	10.1
Monthly								
95	39.4	55.6	5.0	39.1	28.2	62.5	9.3	39.1
96	33.6	62.3	4.1	39.1	35.5	41.8	22.7	39.1
97	35.1	54.1	10.8	39.1	30.5	45.6	24.0	39.1
98	31.2	55.5	13.3	39.1	23.0	57.3	19.6	39.1
99	34.5	39.3	26.3	39.1	28.1	41.7	30.3	39.1

[a] The portfolio risk measure is given by VaR.

[b] The portfolio risk measure is given by CVaR.

Panel A from Table 4 suggests an interesting conclusion for the VaR constraint's results. It seems that for low confidence levels (say 95% - 96%) the results under normality are fairly similar to the empirical distribution results for daily and monthly frequencies. Note that the proportions invested in cash or the values of VaR are slightly higher than the ones presented in Table 3, meaning that, on average, the risk is slightly overestimated. Notwithstanding, for high confidence levels this conclusion is no longer valid: on average, under normality the risk is underestimated for a given level of return. Focusing on the T-student assumption, we can see that the proportions invested in cash or the values of VaR are always higher than in Table 3, suggesting that the risk is overestimated for a given level of return for all the confidence levels considered. Regarding the CVaR constraint, Panel B indicates the same conclusion as in the VaR measure for both distributional assumptions considered. In sum, regardless the risk measure chosen, assuming a distribution for the returns that differs from the historical one will result in underestimation or overestimation of risk.

The shortcomings of choosing another distribution for the returns can also be seen through the behavior of an efficient frontier that incorporates the downside risk measures, which is similar to a mean-variance frontier. The difference lies in the definition of risk, for example an efficient VaR frontier defines risk as the VaR relative to the benchmark return (φ), whereas the mean-variance frontier relies on the standard deviation (σ) as the measure of risk.

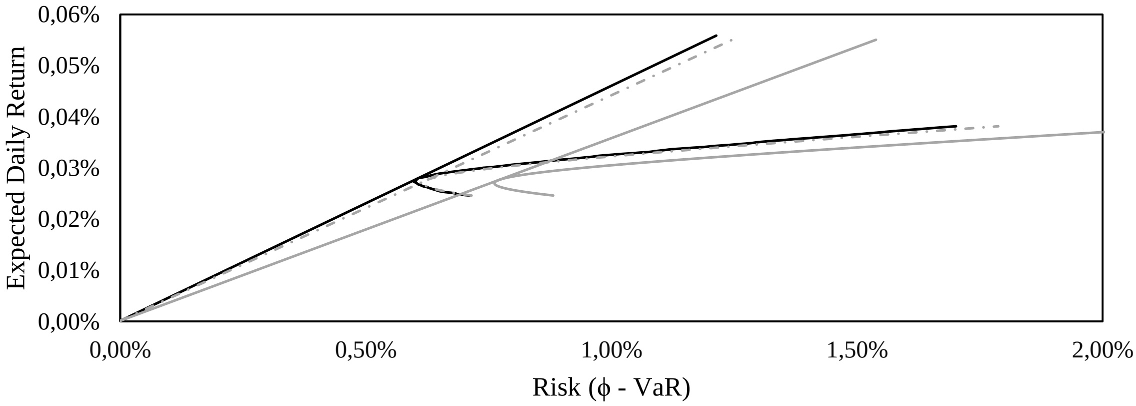
We plotted efficient downside risk frontiers for the 95% and 99% confidence levels under the same alternative distributional assumptions. The empirical distribution illustrates the risk-return trade-off as observed in financial markets, but as the time horizon for the investment increases, the efficient frontier becomes less precise. As a result, we only present the efficient frontiers using daily observations. An investor who wants to be 95% confident that his wealth will not drop by more than the daily threshold chooses the point of the efficient frontier that maximizes the return per unit of risk. The risk-free considered is the last available 3-month Treasury bill rate in the sample period, 0.03%.

In Figure 1 we present the efficient VaR and CVaR frontiers at the 95% confidence level under the empirical, normal and t-student distributions. In order to facilitate the interpretation of results, we present the measure of risk (φ) in the same units as the returns, i.e., in percentage instead of monetary terms. At 95% confidence level we can see that the normal efficient frontier and the normal capital market line are almost identical to the ones of the empirical distribution for both risk measures. Regarding the t-student distribution, it is clear that risk is being overestimated on both downside risk measures: for a given level of return, the risk is higher under the t-student distribution than the empirical distribution.

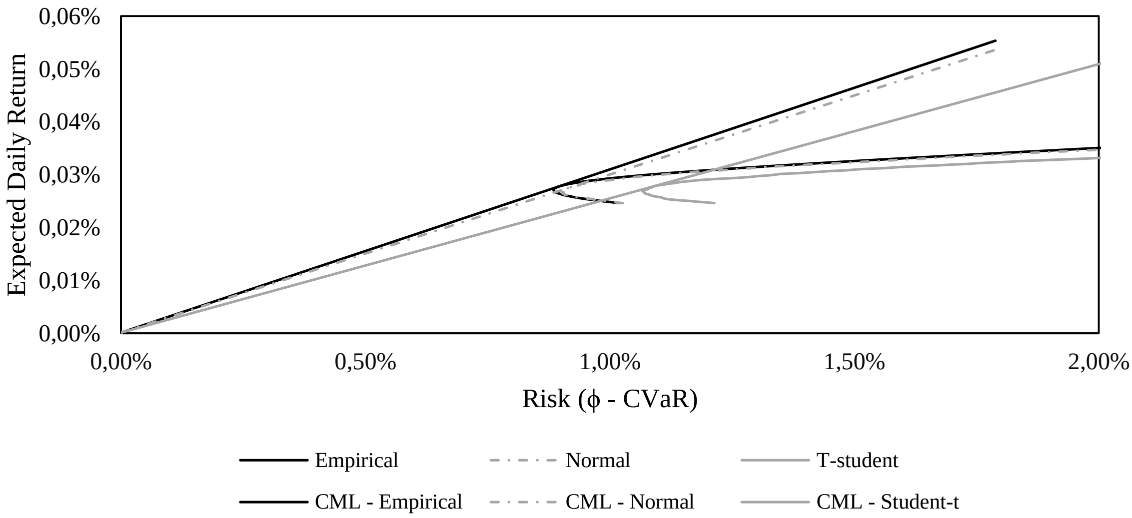
Figure 1
Efficient Downside Risk Frontiers at 95% Confidence Level

This figure presents the risk-return trade-off for portfolios of stocks and bonds, where the risk is considered to be the downside measure ϕ of the portfolio at the 95% confidence level. The frontier is built using daily data on the S&P 500 composite return index and on the 10-year DataStream US benchmark government bond index from February 1988 to July 2014. The risk-free rate considered is the rate on the last period's one month Treasury bill (0.03%). Different distributions are considered: the empirical distribution, the parametric normal approach and the t-student distribution with 5 degrees of freedom.

Panel A: Efficient VaR Frontier



Panel B: Efficient CVaR Frontier

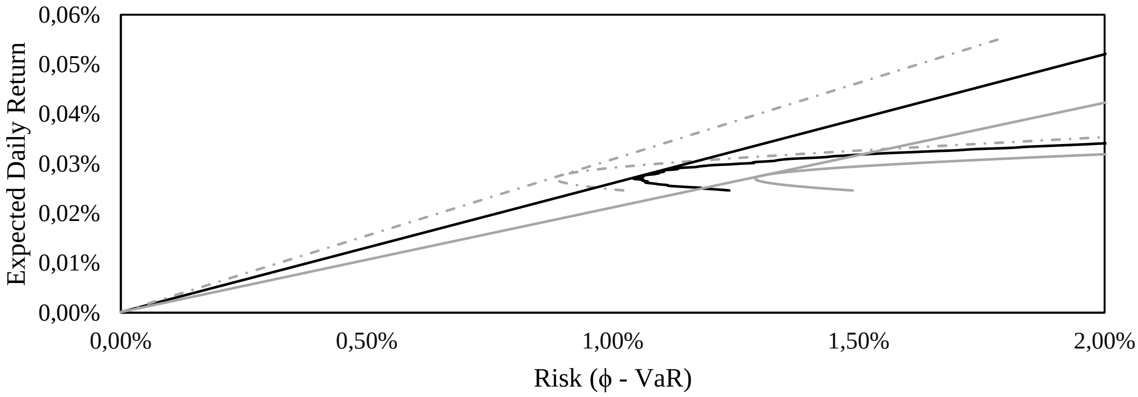


In Figure 2 we present the same efficient frontiers, but now we consider a 99% confidence level. In this case, for a given level of return the risk is underestimated under the normality assumption and overestimated under the t-student distribution.

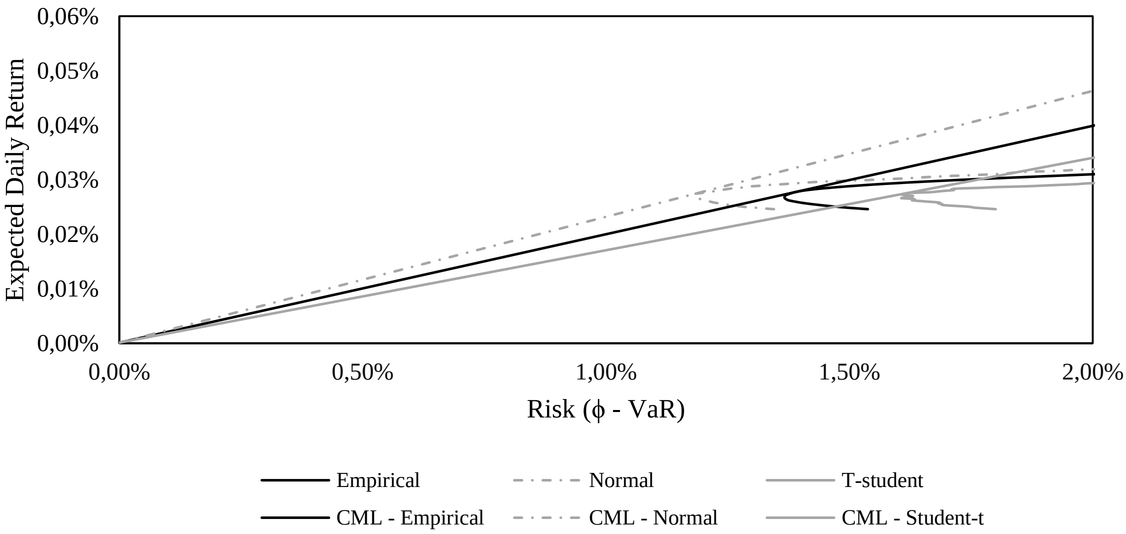
Figure 2
Efficient Downside Risk Frontiers at 99% Confidence Level

This figure presents the risk-return trade-off for portfolios of stocks and bonds, where the risk is considered to be the downside measure ϕ of the portfolio at the 99% confidence level. The frontier is built using daily data on the S&P 500 composite return index and on the 10-year DataStream US benchmark government bond index from February 1988 to July 2014. The risk-free rate considered is the rate on the last period's one month Treasury bill (0.03%). Different distributions are considered: the empirical distribution, the parametric normal approach and the t-student distribution with 5 degrees of freedom.

Panel A: Efficient VaR Frontier



Panel B: Efficient CVaR Frontier



These outcomes lead to the same conclusion from Table 4: for low confidence levels the normal distribution replicates the actual risk-return trade-off fairly well, whereas the t-student distribution results in an overestimation of risk; for high confidence levels the normality assumption tends to underestimate risk, whereas the t-student assumption results in an overestimation of risk. It should be noticed that these conclusions are consistent with the results of Campbell et al. (2001).

5. Performance Evaluation

In this section we compare the performance of downside risk constraint strategies with several benchmarks across time. We also consider the characteristics presented in Section 3 to evaluate whether size and industry have impact or not on the behavior of our strategies. The following portfolios were defined as benchmarks: (1) stock market index, (2) 10-year US benchmark government bond index, (3) Naive portfolio, (4) Sharpe ratio portfolio, and (5) Sortino ratio portfolio. The benchmarks (1) and (2) are the pure risky assets; the portfolio (3) is allocated equally between the two risky assets and the risk-free rate. Portfolio (4) assumes an investor that uses the Sharpe ratio rule. The allocation between the two risky assets is the combination of stocks and bonds that maximizes the Sharpe ratio, which is found at the point where the capital market line is tangent to the efficient frontier. For the allocation between the risky portfolio and the risk-free rate we consider the choice that maximizes a power-utility function with a risk aversion coefficient of 5. The power utility function is given by the following expression:

$$Power\ Utility = \frac{\left(1 + r_p\right)^{1-\gamma}}{1 - \gamma} \tag{11}$$

where r_p is the portfolio return and γ is the risk aversion coefficient. This function takes into account all the moments of the distribution. Regarding the risk aversion coefficient, Engle and Rosenberg (2002) and Tarashev and Tsatsaronis (2006) show that the implicit risk aversion coefficient from option prices is between 2 to 8. We use a risk aversion coefficient of 5 [Brandt (2002), Brandt and Santa Clara (2006)]. The last benchmark, portfolio (5), relies on Sortino ratio as criteria, which is defined as:

$$Sortino\ ratio = \frac{r_p - r_f}{\sqrt{E\left\{Min\left(r_t - r_p, 0\right)^2\right\}}} \tag{12}$$

where, r_p is the average portfolio return and r_t is the portfolio return in day *t*. Therefore, portfolio (5) follows the methodology of portfolio (4), but instead of using the standard deviation as a risk measure, the semi-deviation is used - standard deviation of negative returns.

The performance evaluation consists of an in sample and out of sample exercise using daily data. The portfolios are rebalanced on a monthly basis through a rolling window, using an estimation period of 60 months. In order to study the characteristics effects, the stock market index used changes among the set of stocks indexes presented in Section 3.

5.1. Portfolios' Performance

In Table 5 we present the performance measures and the average weights of the out-of-sample exercise for the benchmarks and for the downside risk constrained models. In the performance measures we include the risk measures VaR and CVaR and the associated performance index of each strategy.

Table 5
Out-of-Sample Performance Statistics

This table shows the OOS returns' performance statistics and the OOS average weights for portfolios (1) to (7). The benchmarks defined are: (1) stocks; (2) bonds; (3) naive portfolio - equally weighted portfolio among stocks, bonds and cash; (4) Sharpe ratio portfolio - for the allocation between the risky portfolio and the risk free rate we use a power utility function with a risk aversion coefficient of 5; (5) Sortino ratio portfolio - similar to Sharpe ratio portfolio, but instead of the standard deviation as measure of risk, it is used the semi-deviation. Portfolios (6) and (7) are found by maximizing the risk-return trade-off from Equation (9) and the decision to borrow or lend is included in the asset allocation exercise. The risk-free rate considered is the rate on the last period's three month Treasury bill (0.03%). The VaR and CVaR for $1000 held in the portfolios are estimated using the historical distribution at the 95% level.

	Benchmarks					Downside Risk Constraints	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	S&P500	Bonds	Naive	Sharpe	Sortino	VaR	CVaR
Returns							
Average return (%)	0.04	0.02	0.02	0.03	0.03	0.03	0.03
Standard deviation (%)	1.19	0.47	0.39	0.61	0.58	0.61	0.67
Minimum (%)	-9.46	-2.83	-3.12	-7.11	-7.11	-7.11	-7.11
Maximum (%)	10.96	4.05	3.66	4.99	4.99	4.99	4.99
Skewness	-9.46	-2.83	-3.12	-7.11	-7.11	-7.11	-0.39
Kurtosis	10.96	4.05	3.66	4.99	4.99	4.99	11.52
VaR ($)	18.35	7.40	5.97	9.18	8.78	9.02	10.74
CVaR ($)	28.76	10.67	8.86	13.75	12.95	14.32	15.85
Ann. Sharpe ratio	0.47	0.78	0.93	0.67	0.70	0.67	0.63
Performance index – VaR	0.19	0.30	0.38	0.28	0.29	0.28	0.25
Performance index – CVaR	0.12	0.21	0.26	0.19	0.20	0.18	0.17
Ann. CE (%)	-0.17	4.51	4.96	4.29	4.48	4.27	3.94
Average Weights							
S&P 500 index	100.0	0.0	33.0	32.2	30.3	32.7	38.6
10-y US Gov Bonds index	0.0	100.0	33.0	67.6	69.5	67.3	61.4
3-month T-bill	0.0	0.0	33.0	0.2	0.2	0.0	0.0

Observing the average weights, there is one straightforward conclusion: when the investor can choose, she prefers to allocate a higher proportion to bonds - portfolios (4) to (7). This might be explained by the fact that we are considering a risk averse investor, meaning that she prefers to invest a higher percentage of her wealth in a less risky asset. Note that since we consider the desired constraints level (95% confidence level) for the downside risk strategies, the investor chooses not invest in the riskless asset.

In terms of risk, it is clear that the naive portfolio is the investment strategy with lowest risk: it shows the lowest values for the standard deviation and the downside risk measures. The explanation behind these results is that naive portfolio is the one with highest proportion invested in the risk-free rate. On the other hand, S&P500 shows the highest values for the standard deviation and the downside risk measures. This is not surprising, since the portfolio is invested 100% in stocks, the riskiest asset in our sample. Regarding the Sharpe, Sortino and downside risk strategies, they present similar values for the risk measures: a standard deviation between 0.58% and 0.67%; a VaR measure between $8 and $11; a CVaR measure between $9 and $16. It should be highlighted that the CVaR strategy presents a significantly lower value for skewness, meaning that extreme negative events are less likely to happen than in the other strategies.

The S&P500 index rewards the investor with the highest daily expected return, 0.04%. The other strategies present an expected return between 0.02% and 0.03%. However, in order to achieve appropriate conclusions, one should analyze the risk-return trade-off. The Naive strategy outperforms the others by rewarding the investor with the highest annualized Sharpe ratio OOS. The other portfolios present an annualized Sharpe ratio between 0.63 and 0.78, except for the S&P500 index, which shows the lowest performance for the risk-return relation. The same conclusion is achieved when taking into account the performance index using VaR or CVaR measures and the annualized certainty equivalent measure.

Considering the results from Table 5 it becomes clear that on average our strategies do not outperform the benchmarks: despite presenting good Sharpe ratios, our strategies yield the worst performance in terms of the certainty equivalent measure. Sharpe and Sortino ratio strategies exhibit a performance very similar to our downside risk constrained models. Notwithstanding, there might be some shortcomings related to these two strategies. First, it is assumed that returns are normally distributed, which might result in underestimation or overestimation of risk as we have seen in the previous section. Second, it is necessary to assume a utility function for the preferences of the investor in order to allocate the portfolio between the riskless and the risky assets. Moreover, if we have not assumed a utility function that also takes into account skewness and kurtosis, probably the results would have been not as good as they are.

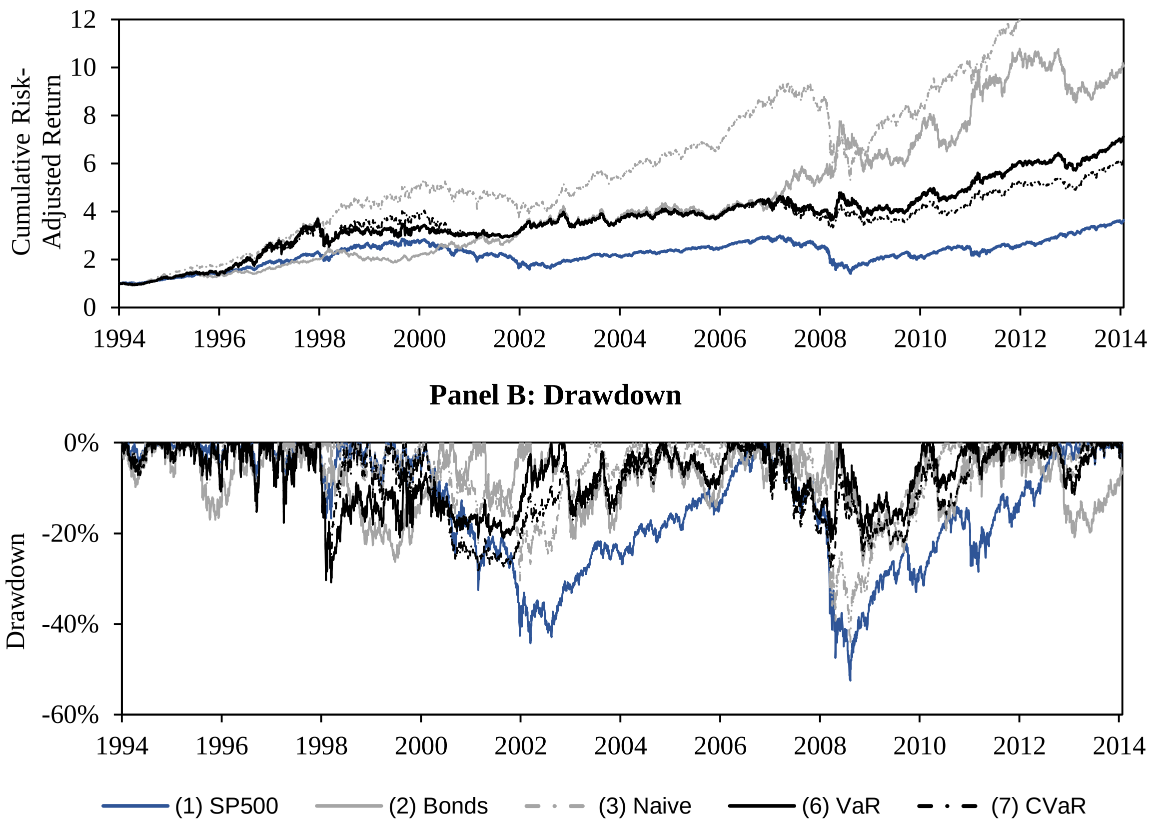
In order to evaluate the performance over time, we present the cumulative risk-adjusted returns and the drawdown for the following portfolios: (1) S&P500, (2) Bonds, (3) Naive, (6) VaR and (7) CVaR. In order to simplify the analysis, we do not present the results for the Sharpe and Sortino ratios, since they are very similar to the downside risk models. The cumulative risk-adjusted returns are calculated by dividing the returns by the standard deviation. In this way, we are taking into account the risk and we are able to compare the performance of all strategies. The drawdown measure is the decline in cumulative returns since a historical peak. For example, a drawdown of -10% in a certain day means that in that day the return of the asset declined 10% in comparison to the last peak verified. The closer to zero the drawdown measure, the better.

Figure 3
Out-of-Sample Performance Across Time

This figure presents the OOS performance of returns across time for the investment period between August 1994 and August 2014. The portfolios presented are the following: (1) stocks; (2) bonds; (3) naive portfolio – equally weighted among stocks, bonds and cash; (6) and (7) are the portfolios formed by the VaR and CVaR constrained strategies, respectively. Panel A shows the cumulative risk-adjusted returns – we divided the returns by the standard deviation in order to be able to compare the different strategies in the graph. Panel B shows the drawdown measure – the percentage decline in cumulative returns since a historical peak. The portfolios are rebalanced on a monthly basis using a rolling window. An estimation period of 60 months is used. In order to maximize the number of observations, we use daily data. The VaR and CVaR portfolios are allocated assuming a 95% confidence level.

Panel A: Cumulative Risk-Adjusted Returns

Panel B: Drawdown



We saw that the S&P 500 index's average risk-adjusted performance is low in comparison to the other portfolios and the same conclusion is reflected in Panel A of Figure 3. The greatest performance of naive portfolio and bonds index is also shown in

Panel A. The VaR and CVaR portfolios show a very similar behavior and achieve a cumulative risk-adjusted return of 7.11 and 6.15, respectively, at the end of the investment period. Over the holding period chosen, the naive strategy rewarded the investor with the highest cumulative risk-adjusted return and *a posteriori* one concludes that the investor should have distributed her wealth equally by the three assets.

Regarding Panel B, one can see that the S&P 500 exhibits the highest drawdowns. Moreover, the other strategies considered show similar drawdowns, except for two periods: 2002 and 2008. In fact, Panel B of Figure 3 presents an interesting pattern. Note that when the S&P 500 index and the naive portfolio are in a trough, their cumulative risk-adjusted performance is lower and their drawdown is deeper, which is normal. However, the bonds index and the downside risk measures strategies show a different pattern: when the market starts to fall, they also show a decreasing trend, but in the next moment they show a higher cumulative risk-adjusted performance and a lower drawdown. In the sample period considered there were two major critical events in the economy: the stock market downturn in 2002 due to the bursting of the information technology bubble; and the subprime mortgage backed securities (MBS) crisis in 2008. As it is possible to see in Figure 3, although the naive portfolio exhibits the best results over time, during these periods it shows a negative performance, whereas the shortfall constrained models and the bonds index show a positive performance. These results suggest a higher ability of VaR and CVaR strategies in adjusting to highly uncertain scenarios, such as profound recessions. As a result, we can look at these models as a way of mitigating risk.

In order to further analyze the validity of this idea, we divided the sample period into expansion and recession periods using NBER's definition.[6] In Table 6 we present this exercise's out-of-sample performance measures for the benchmarks and for the downside risk constrained models. Once more, we also include the risk measures VaR and CVaR and the associated performance index of each strategy.

[6] "A recession is a period between a peak and a trough, and an expansion is a period between a trough and a peak. During a recession, a significant decline in economic activity spreads across the economy and can last from a few months to more than a year. Similarly, during an expansion, economic activity rises substantially, spreads across the economy, and usually lasts for several years." NBER's Business Cycle Dating Committee.

Table 6
Out-of-Sample Performance Statistics During Recession and Expansion Periods

This table shows the OOS returns' performance statistics and the OOS average weights for portfolios (1) to (7) divided by expansion and recession periods. The business cycles were defined using the NBER's classification. Panel A shows the results for expansion periods and the results for recession periods are shown in Panel B. The results for expansion periods are exhibited in Panel A and The benchmarks defined are: (1) stocks; (2) bonds; (3) naive portfolio - equally weighted portfolio among stocks, bonds and cash; (4) Sharpe ratio portfolio - for the allocation between the risky portfolio and the risk free rate we use a power utility function with a risk aversion coefficient of 5; (5) Sortino ratio portfolio - similar to Sharpe ratio portfolio, but instead of the standard deviation as measure of risk, it is used the semi-deviation. Portfolios (6) and (7) are found by maximizing the risk-return trade-off from Equation (9) and the decision to borrow or lend is included in the asset allocation exercise. The risk-free rate considered is the rate on the last period's three month Treasury bill (0.03%). The VaR and CVaR for $1000 held in the portfolios are estimated using the historical distribution at the 95% level.

Panel A: Expansion Periods

	Benchmarks					Downside Risk Constraints	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	S&P500	Bonds	Naive	Sharpe	Sortino	VaR	CVaR
Av. return (%)	0.05	0.02	0.03	0.03	0.03	0.03	0.03
Std. deviation (%)	1.02	0.44	0.34	0.61	0.58	0.61	0.66
VaR ($)	16.30	7.04	5.40	8.93	8.47	8.81	10.25
CVaR ($)	23.98	10.06	7.53	11.74	10.77	14.32	15.70
Performance index - VaR	0.32	0.30	0.51	0.33	0.35	0.34	0.31
Performance index - CVaR	0.22	0.21	0.37	0.25	0.28	0.21	0.20
Ann. Sharpe ratio	0.81	0.79	1.33	0.78	0.82	0.78	0.78
Ann. CE (%)	6.81	4.41	6.66	5.37	5.60	5.40	7.33

Panel B: Recession Periods

	Benchmarks					Downside Risk Constraints	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	S&P500	Bonds	Naive	Sharpe	Sortino	VaR	CVaR
Av. return (%)	-0.10	0.03	-0.02	-0.01	-0.01	-0.01	-0.02
Std. deviation (%)	2.15	0.66	0.67	0.63	0.62	0.63	0.75
VaR ($)	35.02	10.60	9.99	10.60	10.41	10.75	12.89
CVaR ($)	54.11	13.89	16.44	16.66	16.44	14.04	16.35
Performance index - VaR	-0.30	0.29	-0.22	-0.07	-0.08	-0.08	-0.16
Performance index - CVaR	-0.19	0.22	-0.13	-0.04	-0.05	-0.06	-0.13
Ann. Sharpe ratio	-0.77	0.75	-0.52	-0.19	-0.20	-0.22	-0.44
Ann. CE (%)	-42.73	5.29	-8.02	-4.26	-4.31	-4.59	-6.41

In Panel A of Table 6, it is possible to see that all the portfolios show a very good performance during expansions, as expected. According to the risk return trade-off, the naive portfolio yields the best results, but the certainty equivalent measure reveals the stock market index as the best strategy. The shortfall constrained models show a similar performance to the Sharpe and Sortino ratios and also to the government bonds index. Regarding the results presented in Panel B, notice that during recessions the 10-year U.S.

Government Bonds Index turns out to be the best portfolio. It is the only strategy with positive values for the performance measures: a Sharpe ratio of 0.75, a performance index with VaR/CVaR of 0.29 and 0.22, respectively and a certainty equivalent of 8.21%. Despite all the other portfolios presenting negative performance measures, it is clear that the S&P500 and the naive portfolios yield the worst results. Furthermore, from Panel B one can also conclude that the VaR strategy seems to suffer less than the CVaR strategy during recessions.

In conclusion, when considering the whole sample period, the downside risk constrained models show good results out of sample, but do not outperform all the benchmarks. Notwithstanding, by splitting the sample into expansion and recession periods we are able to conclude that the best strategy would be to invest in the naive portfolio during expansions and in the 10-year U.S. Government Bonds during recessions. In fact, if the investor is able to implement this strategy under perfect timing, she would be rewarded with an annualized Sharpe ratio of 1.17. However, the investor is not capable to perfectly anticipate the periods of economic peaks and troughs and, therefore, this optimal strategy cannot be implemented. The downside risk constraints models may represent a solution to this problem. They reward the investor with good results during expansions and reasonably good results during recessions. We can say that these models satisfy the needs of risk averse investors by mitigating the risk. Furthermore, they do not require any distributional assumptions for the returns or utility functions for the investor's preferences as the Sharpe and Sortino ratios do. Therefore, we are able to evade several shortcomings and to achieve more accurate results.

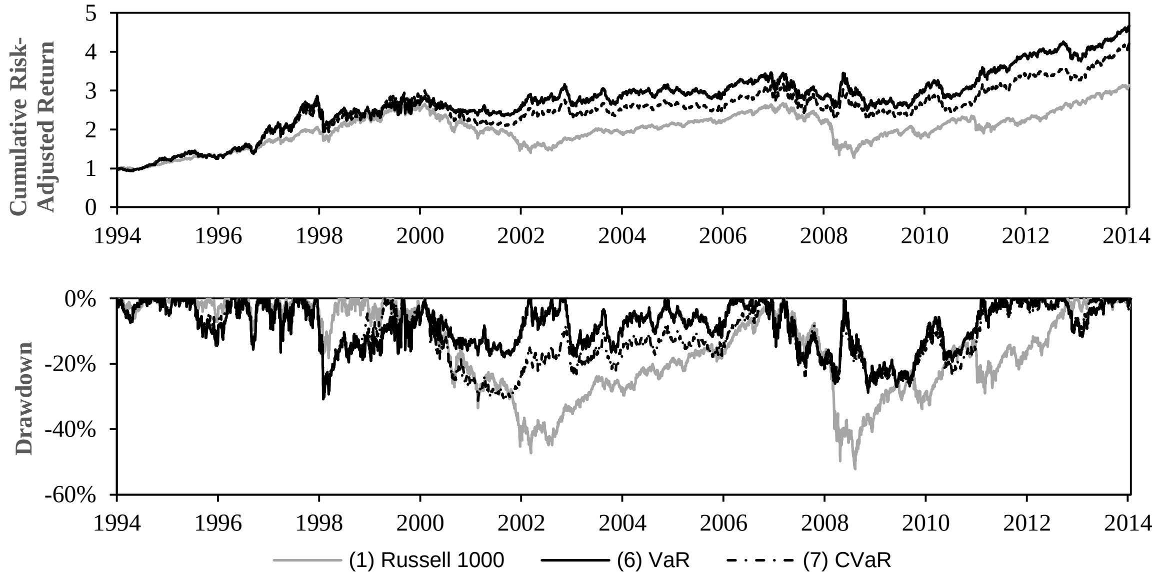
5.2. Characteristics vs Portfolios' Performance

In the previous sections we have allocated the portfolio among stocks, bonds and cash. Now, we follow the same methodology, but we use different proxies for stocks. The purpose of doing so is to study whether the results achieved in Section 5.1 are consistent or not when considering different market conditions. Therefore, we consider two different characteristics: size and industry. The proxies for size are the Russell 1000 and the Russell 2000 indexes. For the industry characteristic we use the S&P 500 Energy, the S&P 500 Financials and the S&P 500 Software indexes. The results are shown in Figure 4 for size and in Figure 5 for industry.

Figure 4
Size Characteristic: OOS Performance

This figure presents the OOS cumulative risk-adjusted returns and their drawdown over the investment period August 1994 - August 2014 for the size characteristic. Panel A focus on large market capitalization, whereas Panel B focus on small market capitalization. The results are shown for Portfolios (1) – the stock market index; (6) and (7) – the downside risk constraints models at 95% confidence level. The VaR and CVaR portfolios are composed of one stock index (Russell 1000 or Russell 2000), the 10-year U.S. government bonds index and the risk free rate. The portfolios are rebalanced on a monthly basis using a rolling window. An estimation period of 60 months is used. In order to maximize the number of observations, we use daily data.

Panel A: Large Cap



Panel B: Small Cap

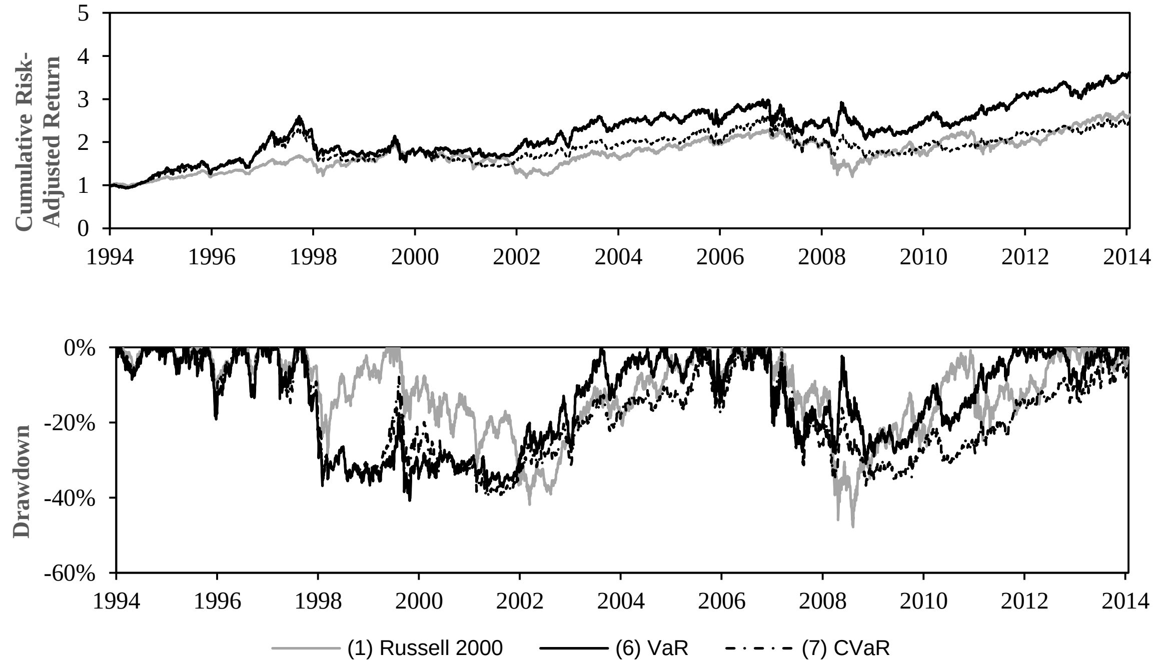
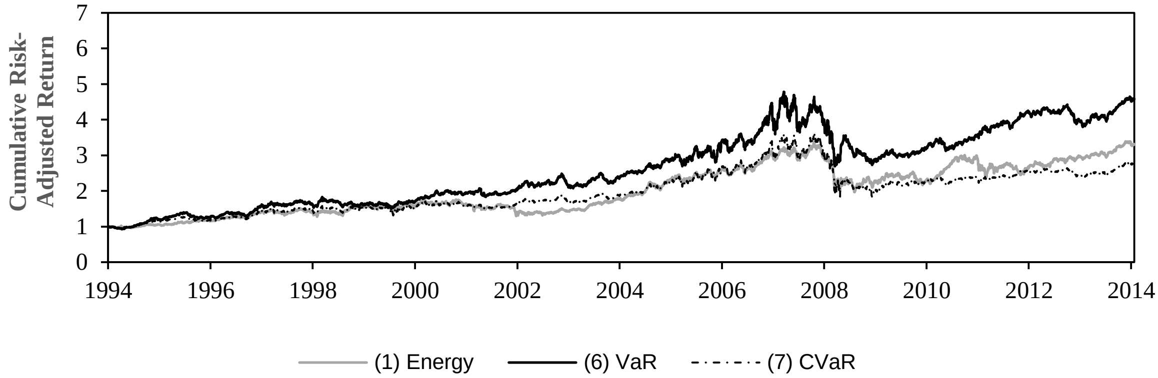


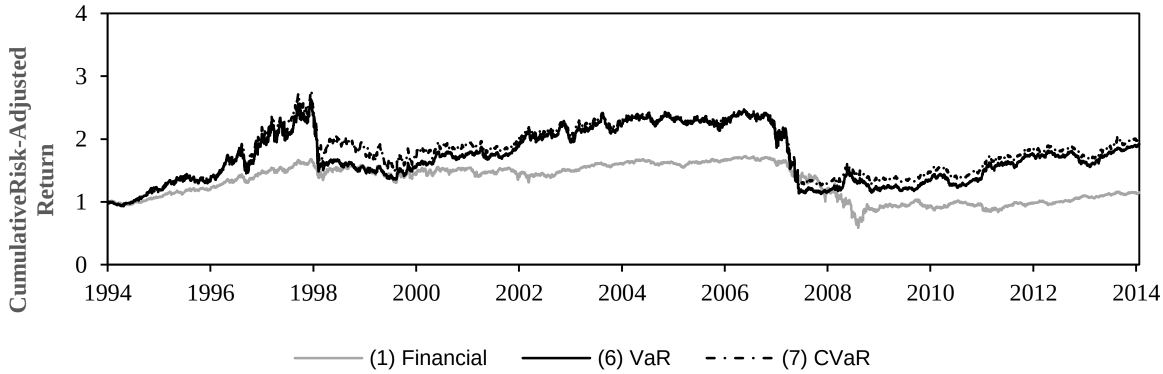
Figure 5
Industry Characteristic: Cumulative Risk-Adjusted Returns

This figure presents the OOS cumulative risk-adjusted returns over the investment period August 1994 - August 2014 for the industry characteristic. Panel A focus on energy, Panel B on financials and Panel C on technology. The results are shown for Portfolios (1) – the stock market index; (6) and (7) – the downside risk constraints models at 95% confidence level. The VaR and CVaR portfolios are composed of one stock index (S&P 500 Energy, S&P 500 Financials or S&P 500 Software), the 10-year U.S. government bonds index and the risk free rate. The portfolios are rebalanced on a monthly basis using a rolling window. An estimation period of 60 months is used. In order to maximize the number of observations, we use daily data.

Panel A: Energy Industry



Panel B: Financial Industry



Panel C: Technology Industry

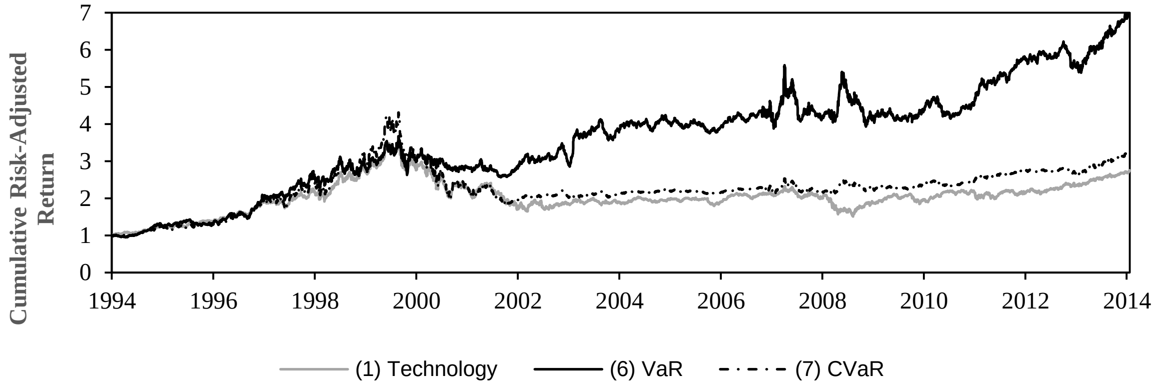
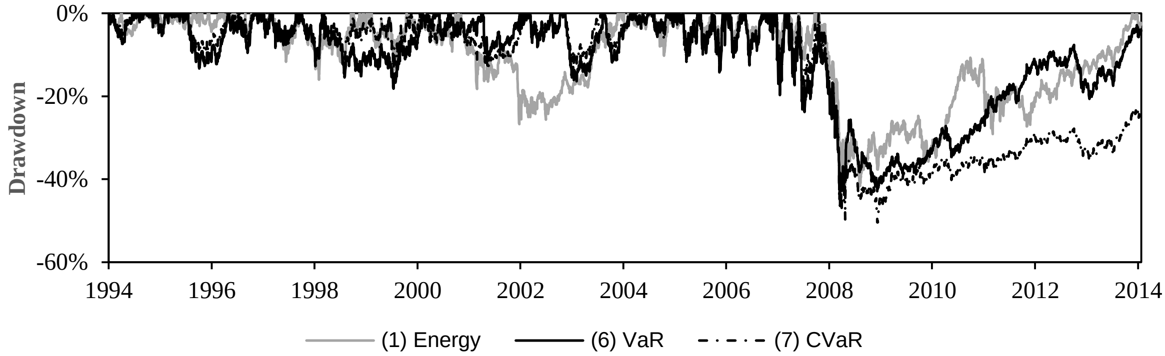


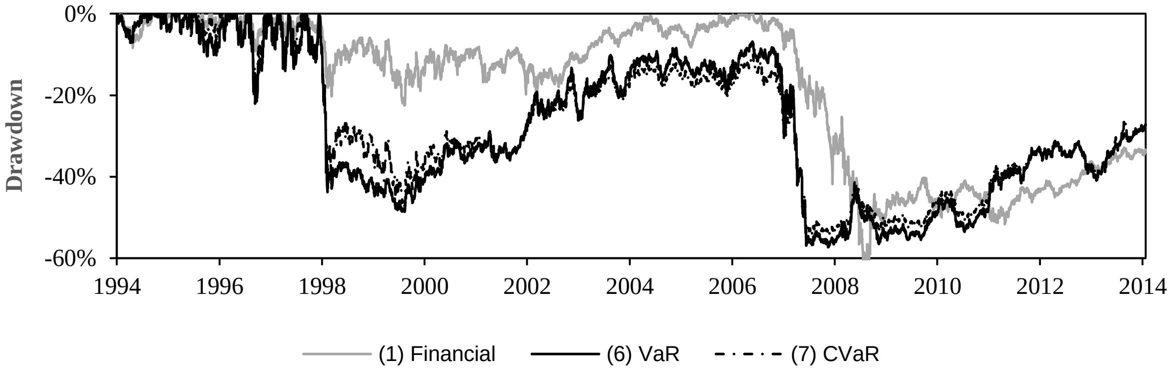
Figure 6
Industry Characteristic: Drawdown

This figure presents the OOS drawdown of cumulative risk-adjusted returns over the investment period August 1994 - August 2014 for the industry characteristic. Panel A focus on energy, Panel B on financials and Panel C on technology. The results are shown for Portfolios (1) – the stock market index; (6) and (7) – the downside risk constraints models at 95% confidence level. The VaR and CVaR portfolios are composed of one stock index (S&P 500 Energy, S&P 500 Financials or S&P 500 Software), the 10-year U.S. government bonds index and the risk free rate. The portfolios are rebalanced on a monthly basis using a rolling window. An estimation period of 60 months is used. In order to maximize the number of observations, we use daily data.

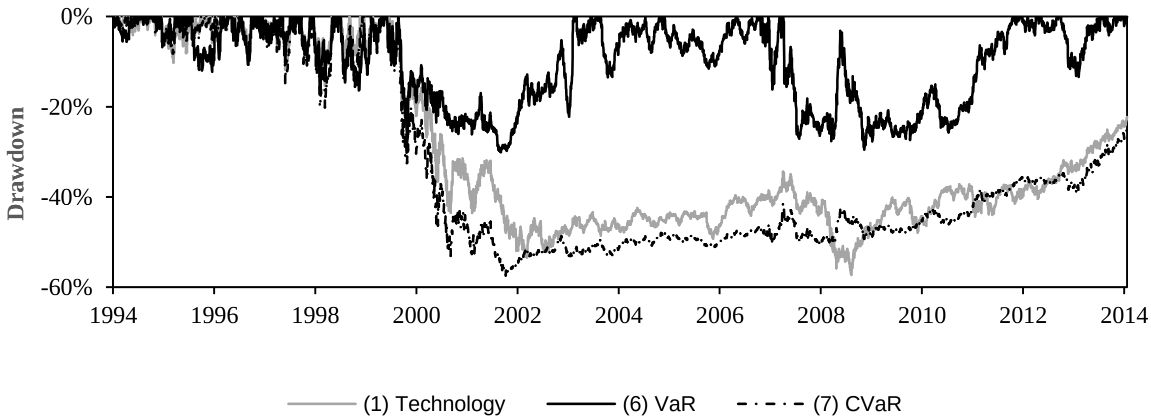
Panel A: Energy Industry



Panel B: Financial Industry



Panel C: Technology Industry



Regarding the size characteristic, one can see from Figure 4 that the downside risk constrained strategies show a better cumulative risk-adjusted performance for both large and small cap. Moreover, when focusing on the financial crisis of 2008 results, one concludes that the stocks market index exhibits a poor performance whereas our strategies show the opposite: the cumulative risk-adjusted return increases and the drawdown is much lower. Taking into account these results, we can say that the size characteristic does not have impact on the behavior of our models.

When focusing on the industry characteristic there are some events that should be noticed. First, all the industries were affected by the financial crisis of 2008, which is not surprising. Second, in Panels C from Figures 5 and 6 we can see clearly the effects of the technological bubble of 2000: the technology index was severely affected, showing a maximum drawdown of -53% during this period. An interesting fact is that the CVaR strategy tends to follow the pattern of the stock market index, whereas the VaR strategy continues to yield the best performance. This is the first time that the evolution of our strategies diverge. Finally, the last event leads to the same conclusion as before: during recession periods our strategies outperform the stock market indexes.

There is an interesting conclusion from both figures. As before, we can see that when the market is entering into a trough our strategies also show a decreasing trend. However, immediately after, as the market is still tumbling, the downside risk constrained strategies show an increasing trend - a higher cumulative risk-adjusted performance and a lower drawdown. These results are easily seen during the financial crisis of 2008. There is evidence that these models exhibit a fast recovery during economic downturns, meaning that the investor is able to mitigating risk by investing under these strategies.

In sum, there are two important conclusions regarding the characteristics exercise. First, both risk measures exhibit a similar performance. Second, none of the characteristics seem to have impact in the performance of our strategy.

6. Concluding Remarks

We focus on downside risk as an alternative measure for risk in financial markets. Following the methodology of Campbell et al. (2001) we develop a framework for portfolio optimization that moves away from the standard mean-variance selection and does not require distributional assumptions for the returns. We define risk in terms of the downside risk measure above the risk free rate on the initial wealth. We use VaR and CVaR as downside risk measures, which enables us to illustrate the investor's level of risk aversion through the confidence level associated with each measure. The optimal portfolio is found by maximizing the expected return subject to the level of risk.

We consider different distributional assumptions for the returns (historical, normal and t-student distributions) and evaluate the impact of non-normal returns. We find that for the VaR at 95% confidence level, the normality assumption reflects the actual risk-return trade-off fairly well. However, as the confidence level increases, the results show an underestimation of risk. The CVaR constraint shows that under normality the risk is overestimated for lower confidence levels and underestimated for higher confidence levels. The assumption of a t-student distribution results in a highly overestimation of risk for both risk measures. Therefore, regardless the risk measure considered, assuming a distribution for the returns that differs from the historical one will result in underestimation or overestimation of risk.

When evaluating the performance over time, one concludes that the downside risk constrained models outperform the market, but not the other benchmarks considered. We find that the optimal strategy would be to invest in the naive rule during expansions and in the bonds index during recessions, but the investor is not able to fully predict the periods of economic peaks and troughs. The downside risk constrained models may yield a good alternative: the investor is rewarded with good results during expansions and reasonably good results during recessions. Furthermore, these models do not require any distributional assumptions or utility functions as the Sharpe and Sortino ratios do. Therefore, we are able to evade several shortcomings and to achieve more accurate results. These results are robust when taking into consideration the size and industry characteristics. In conclusion, a more risk averse investor would prefer to apply a downside risk constraint and follow our strategy in order to incur less risk.

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